Fourier Transforms

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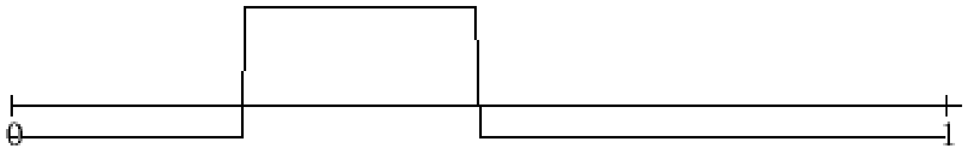
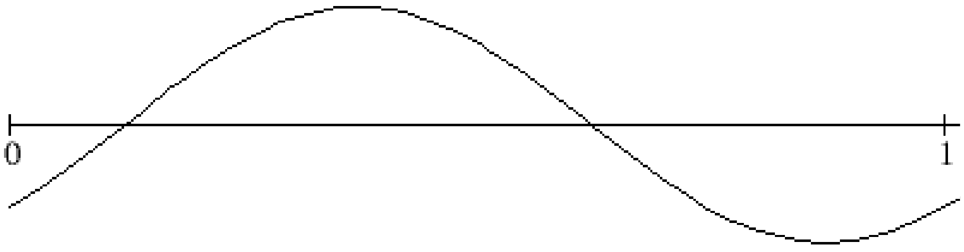
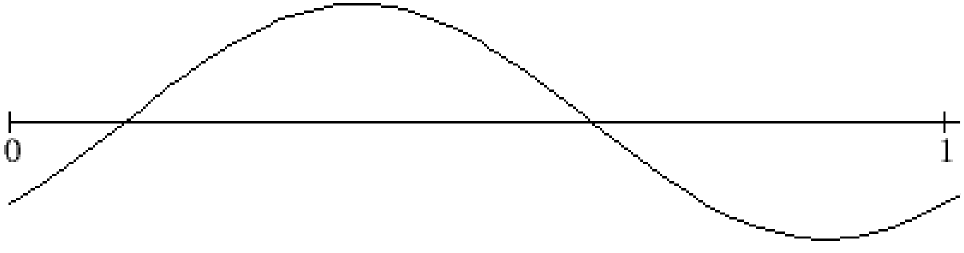
ANY function f(x) can be represented exactly as a sum of sin() functions with specific amplitudes and phases.

Fourier Representation

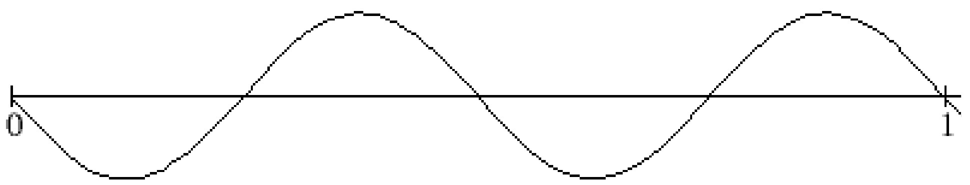


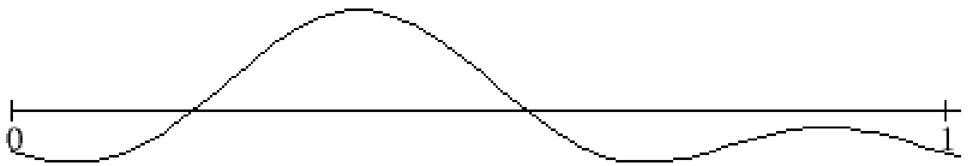
Fourier Representation



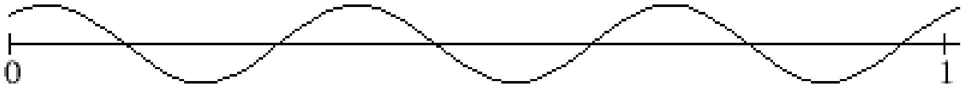


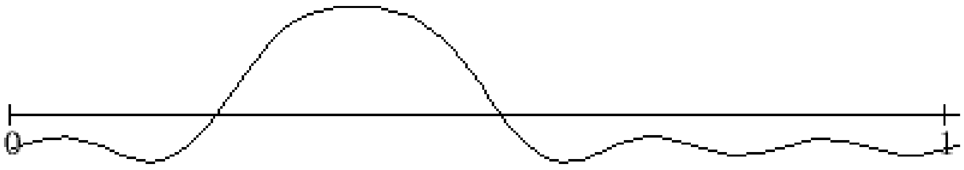
Fourier Representation



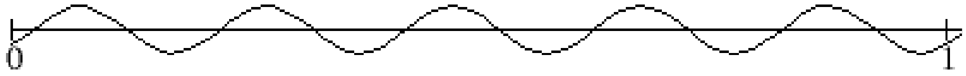


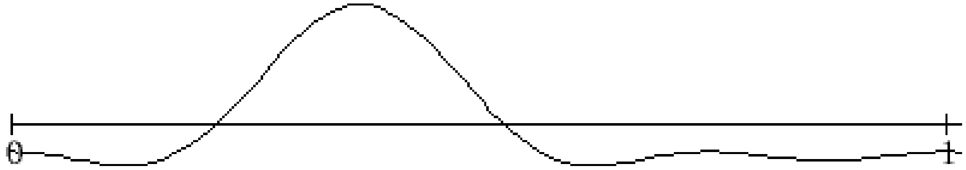
Fourier Representation





Fourier Representation





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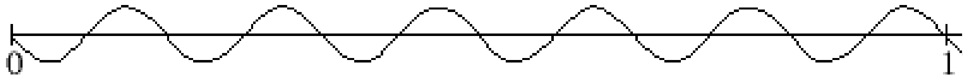
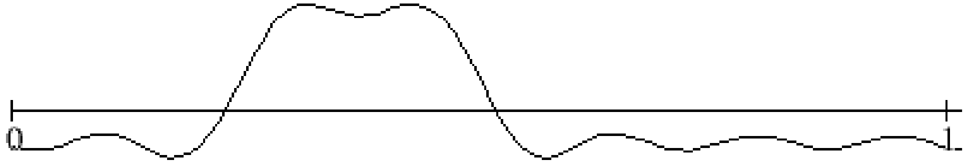
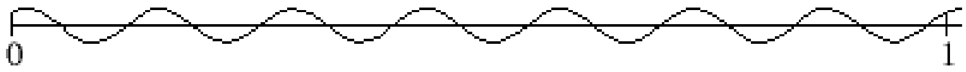
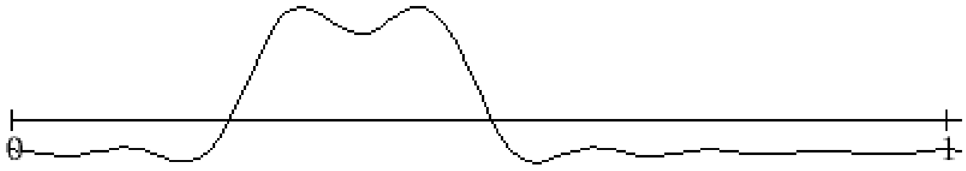


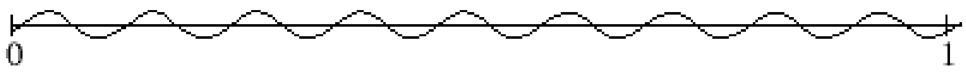


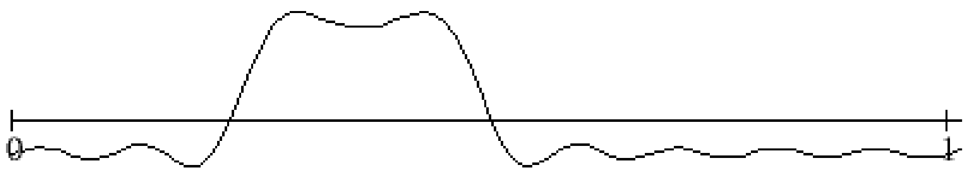
Fourier Representation



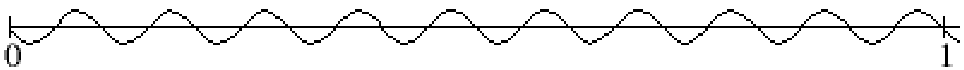


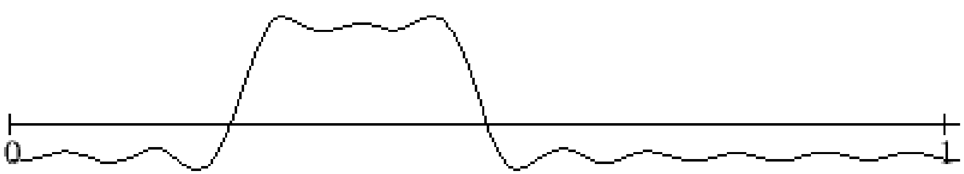
Fourier Representation



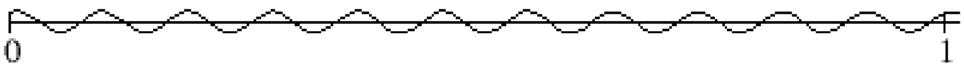


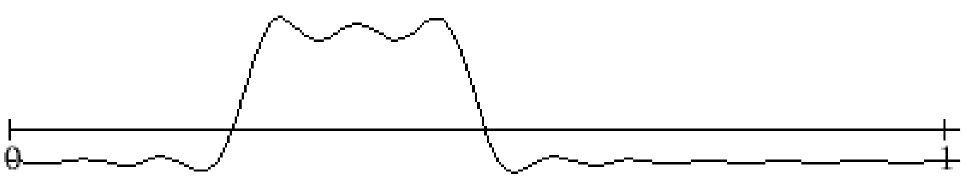
Fourier Representation



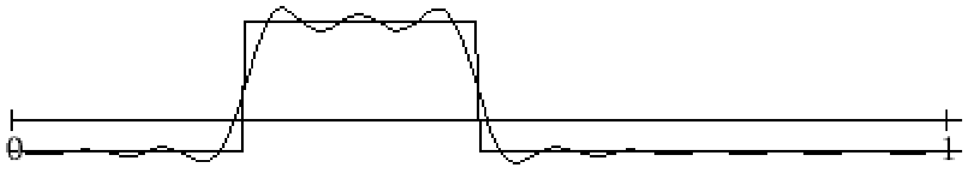


Fourier Representation





Fourier Representation



Fourier Transforms

- We need to know G(k)

$$\int_{-\infty}^{\infty} G(k)\sin(kx)\,dx = f(x)$$

- What about at x=0 ? (completeness)

$$\int_{-\infty}^{\infty} G(k)\sin(kx) + H(k)\cos(kx)\,dx = f(x) \qquad \text{or}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{F}(k)\,e^{ikx}\,dx = f(x)$$

- How do we get F(k)? (orthogonality)

$$\int_{-\infty}^{\infty} e^{ik_1 x} e^{ik_2 x}\,dx = 0 \qquad (k_1 \neq k_2)$$

Fourier Transforms

$$\bar{F}(k)=\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$

$$e^{ikx}=\cos(kx)+i\sin(kx)\rightarrow$$

$$\bar{F}(k)=\int_{-\infty}^{\infty} f(x)\cos(kx)\,dx-i\int_{-\infty}^{\infty} f(x)\sin(kx)\,dx$$

$$a(k)+ib(k)\ \text{ - or - }\ \sqrt{a(k)^2+b(k)^2},\ \tan^{-1}\frac{b(k)}{a(k)}$$

Fourier Transforms

$$\bar{F}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{F}(k) e^{ikx} dk$$

Fourier Transform Theorems

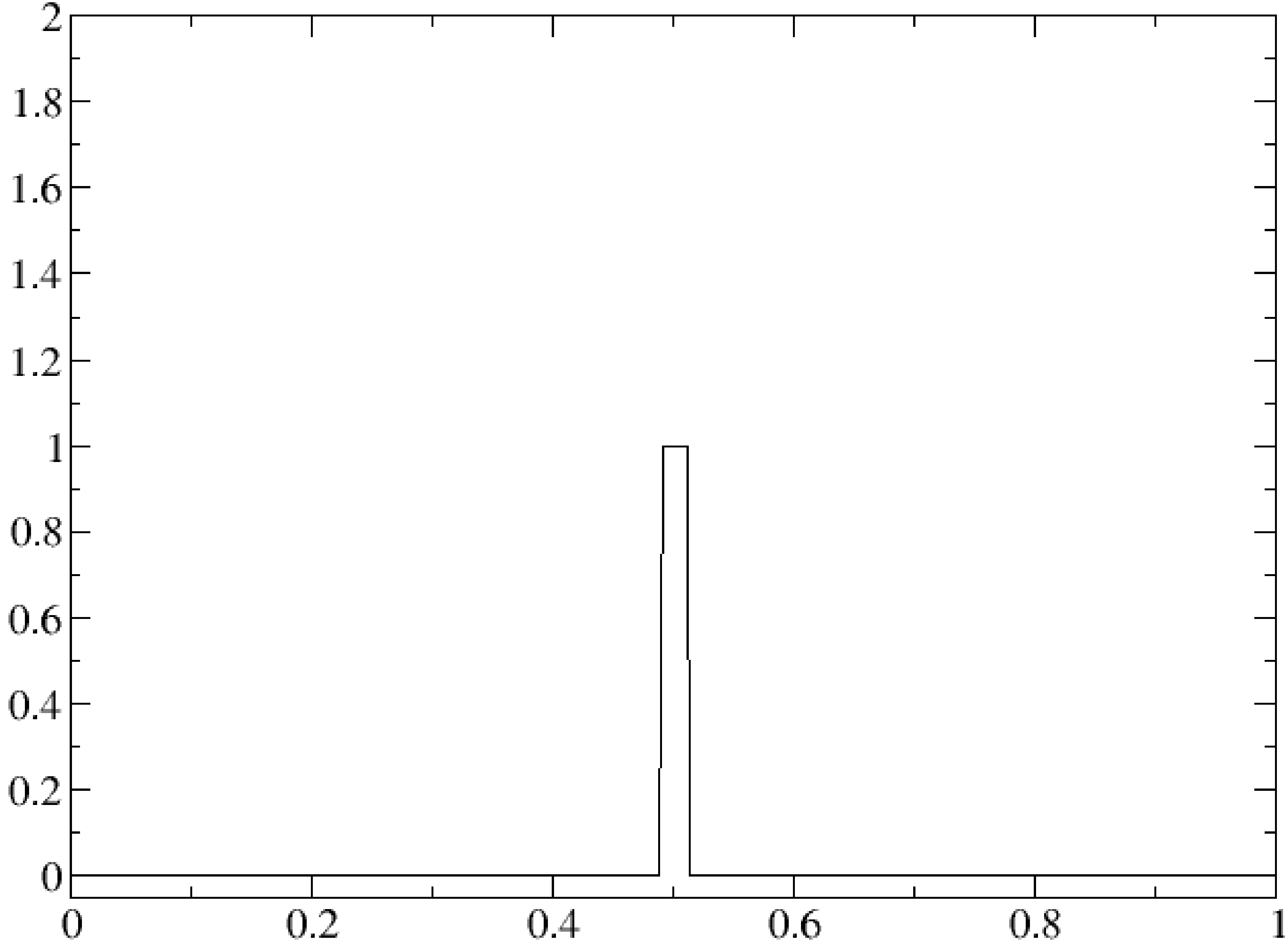
$$\text{if } f(x) \text{ real} \rightarrow \bar{F}(k) = \bar{F}^{*}(-k)$$

$$Convolution: f * g \rightarrow \bar{F}(k)\bar{G}(k)$$

$$Correlation: \int_{-\infty}^{\infty} g(x+a)h(a)\,da \rightarrow \bar{G}(k)\bar{H}^{*}(k)$$

$$Translation: f(x+x_0) \rightarrow \bar{F}(k)e^{ikx_0}$$

FFT of a Square Pulse



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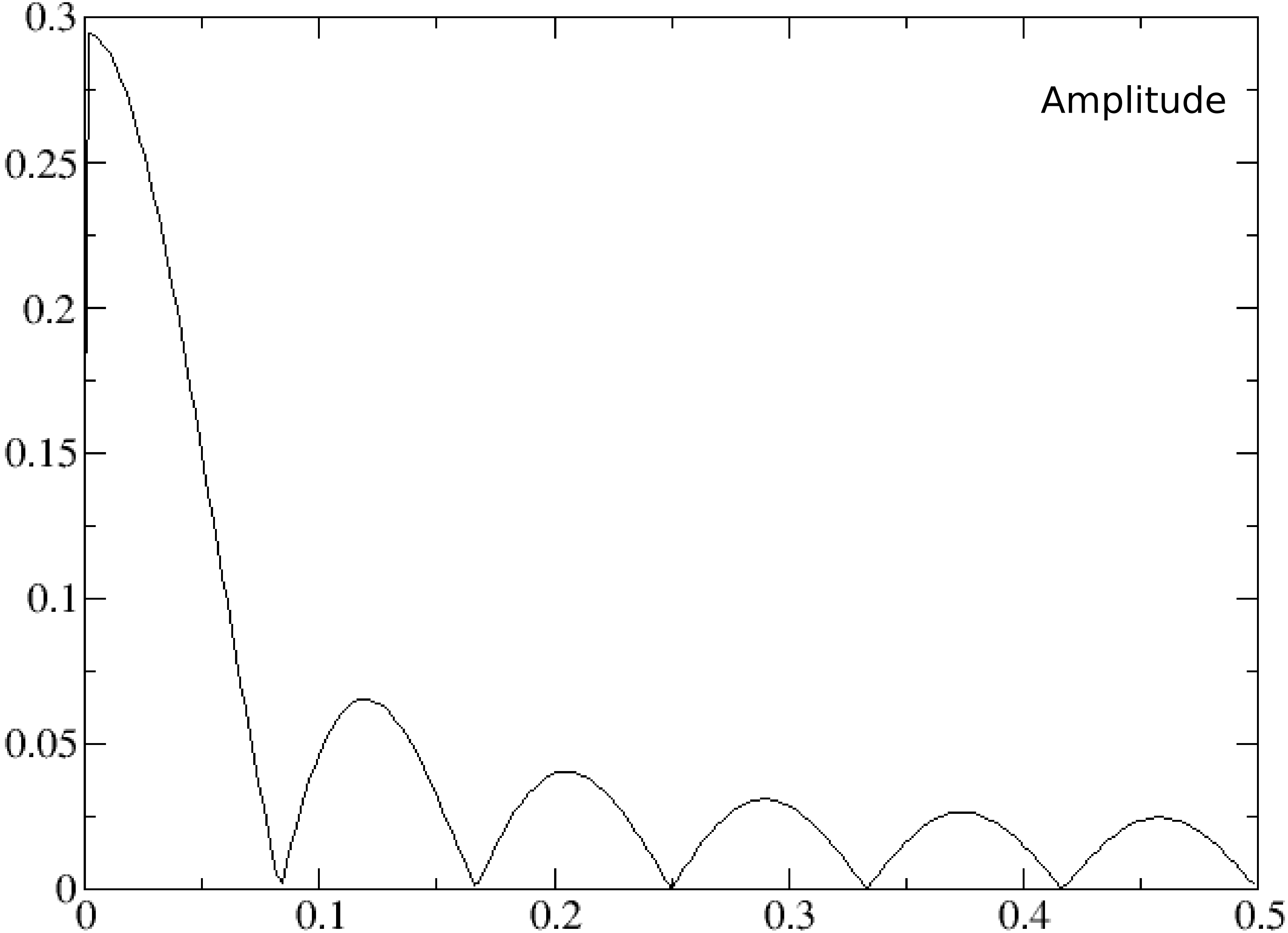
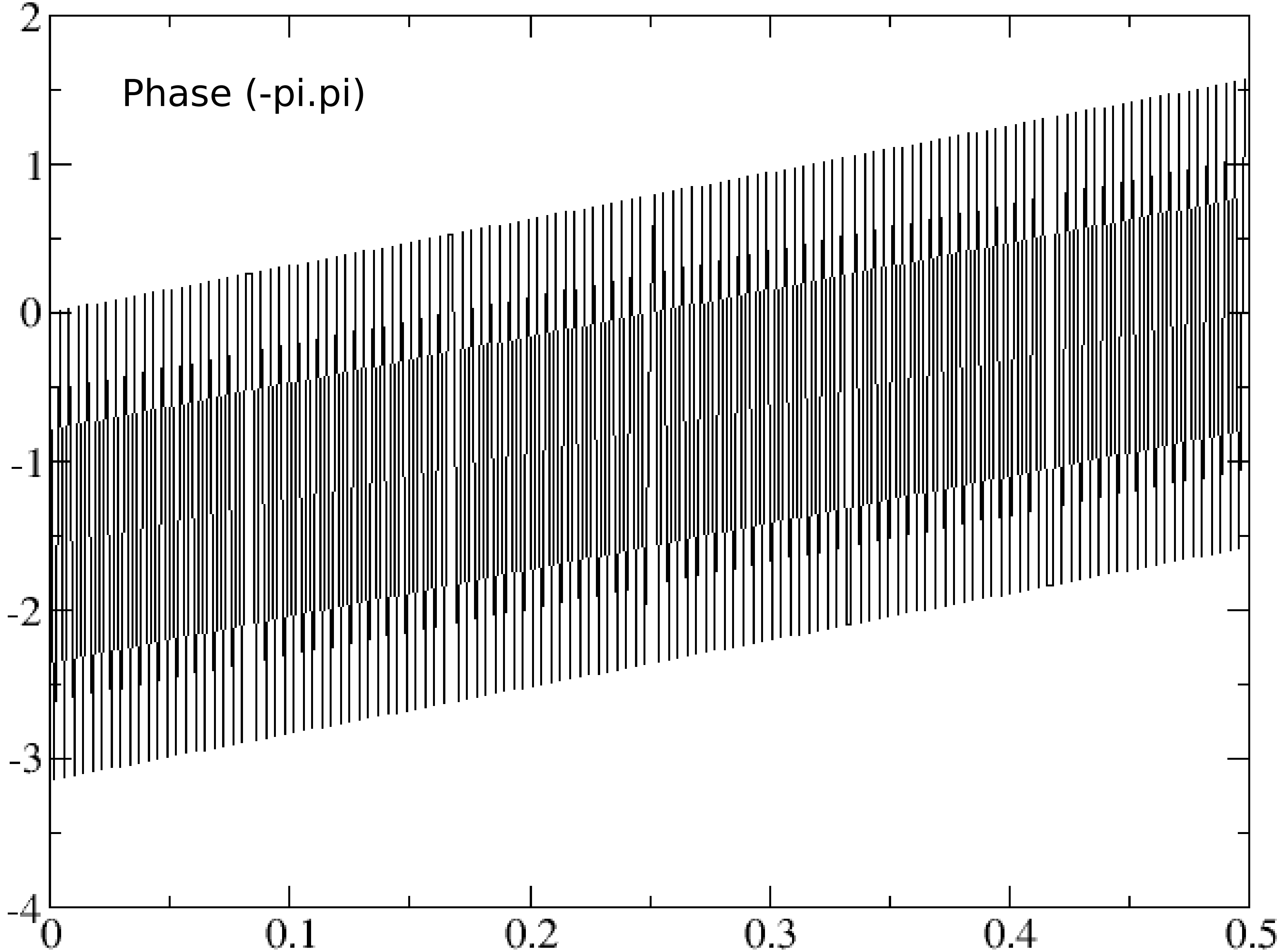
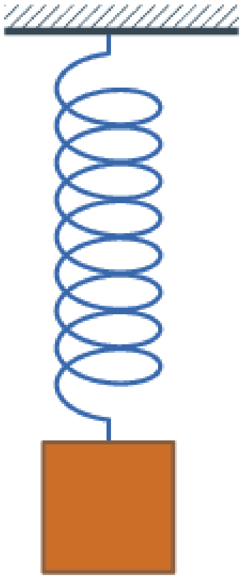


FFT of a Square Pulse



Resonance

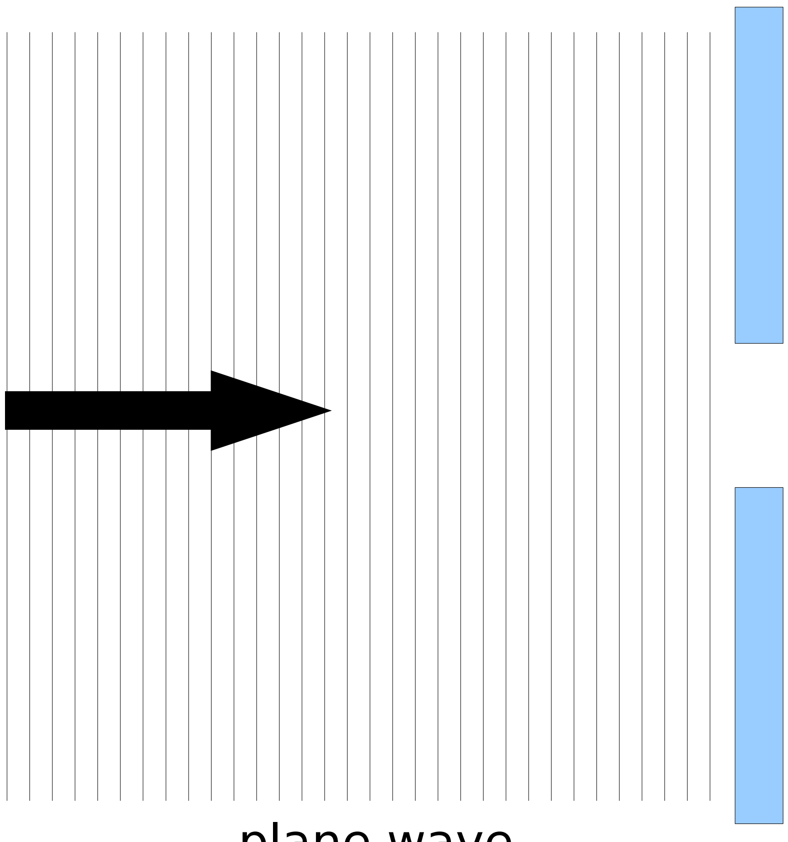
- LC circuit (radio tuner)
- Musical instrument
- Harmonic oscillator



Optics

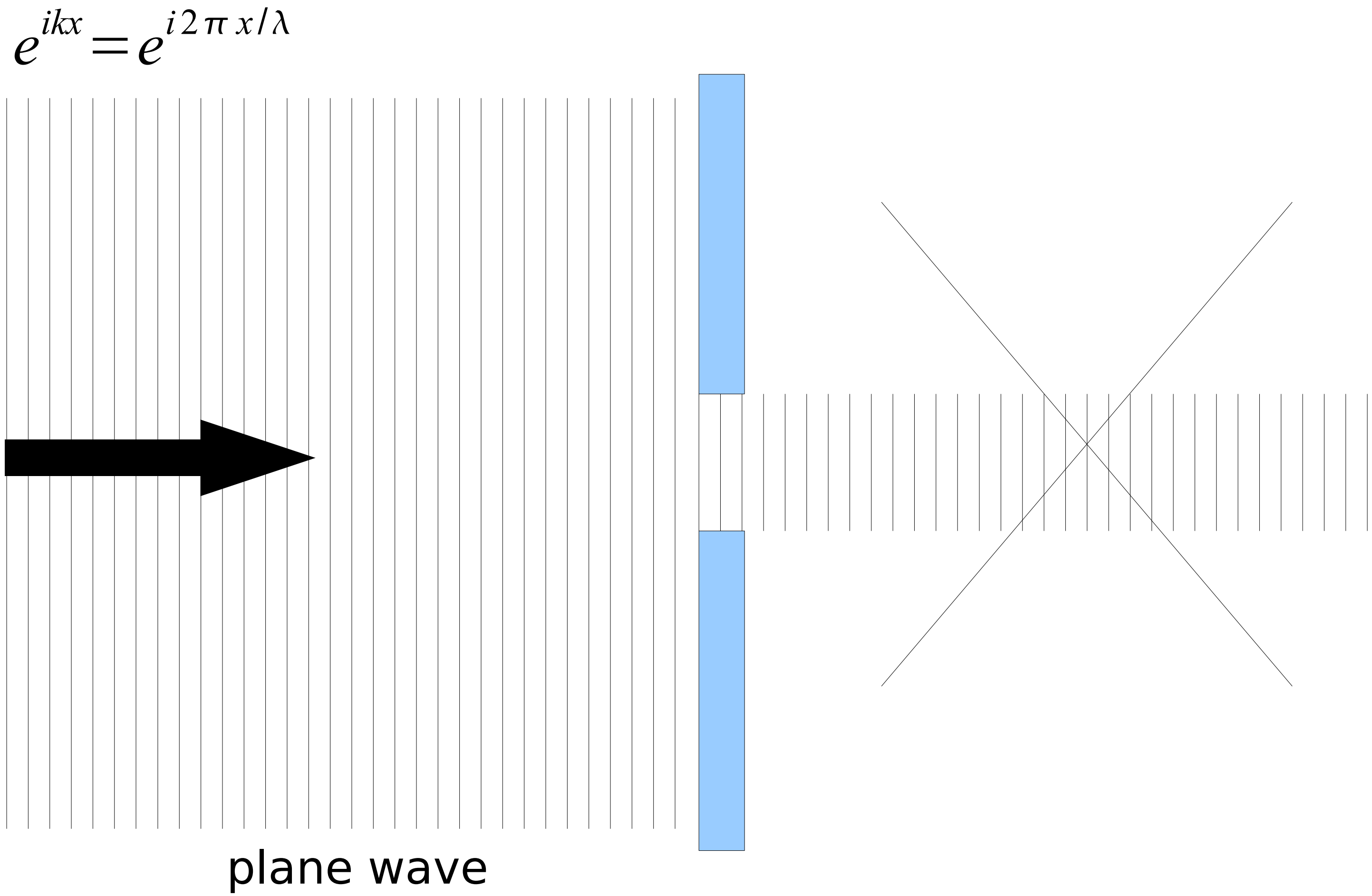
Single Slit Experiment

$e^{ikx} = e^{i2\pi x/\lambda}$



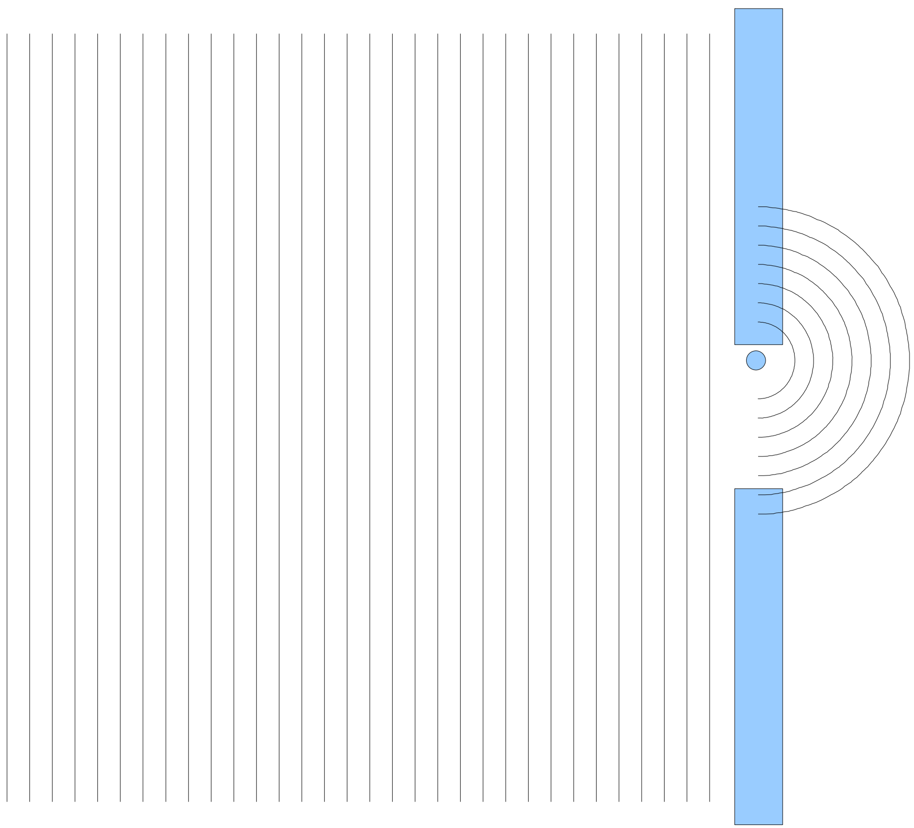
plane wave

Single Slit Experiment



Single Slit Experiment

$$e^{ikx} = e^{i2\pi x/\lambda}$$



Single Slit Experiment

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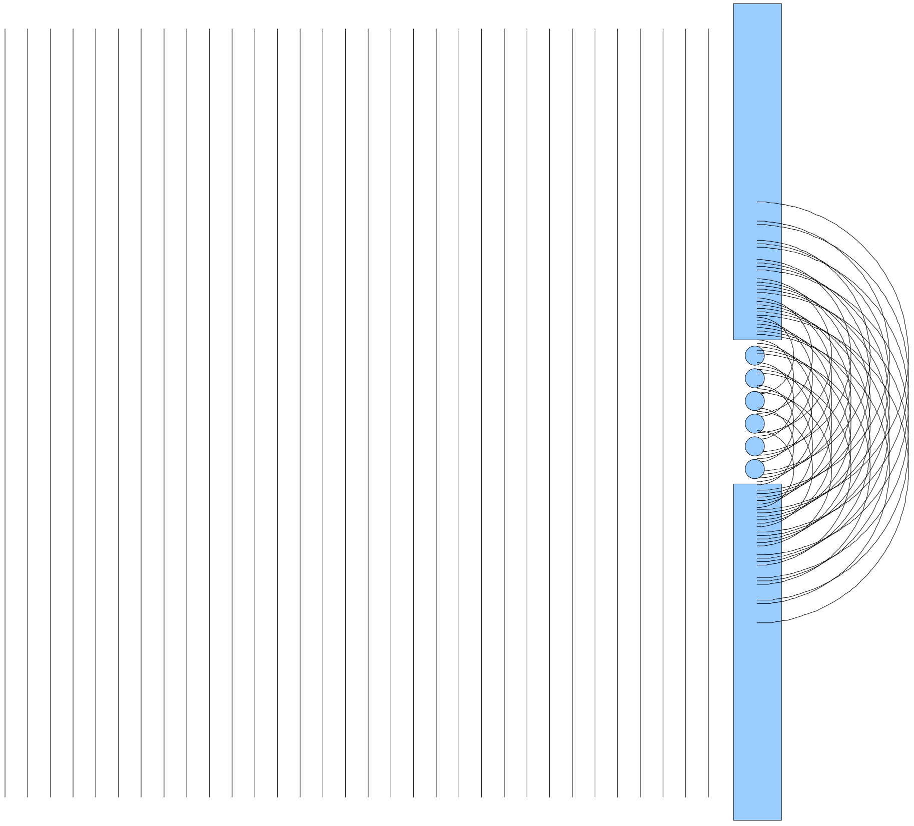

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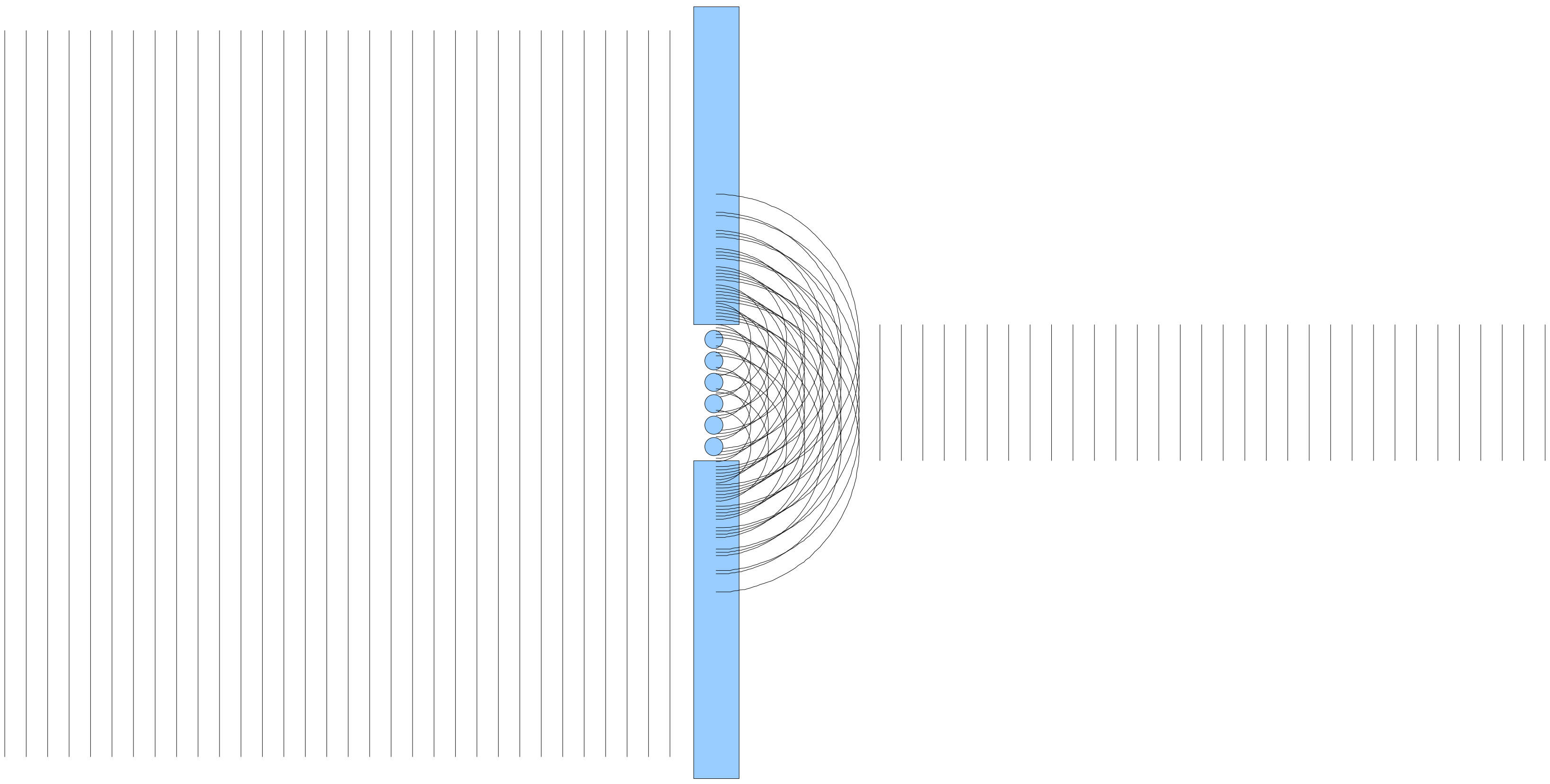
$$e^{ikx} = e^{i2\pi x/\lambda}$$



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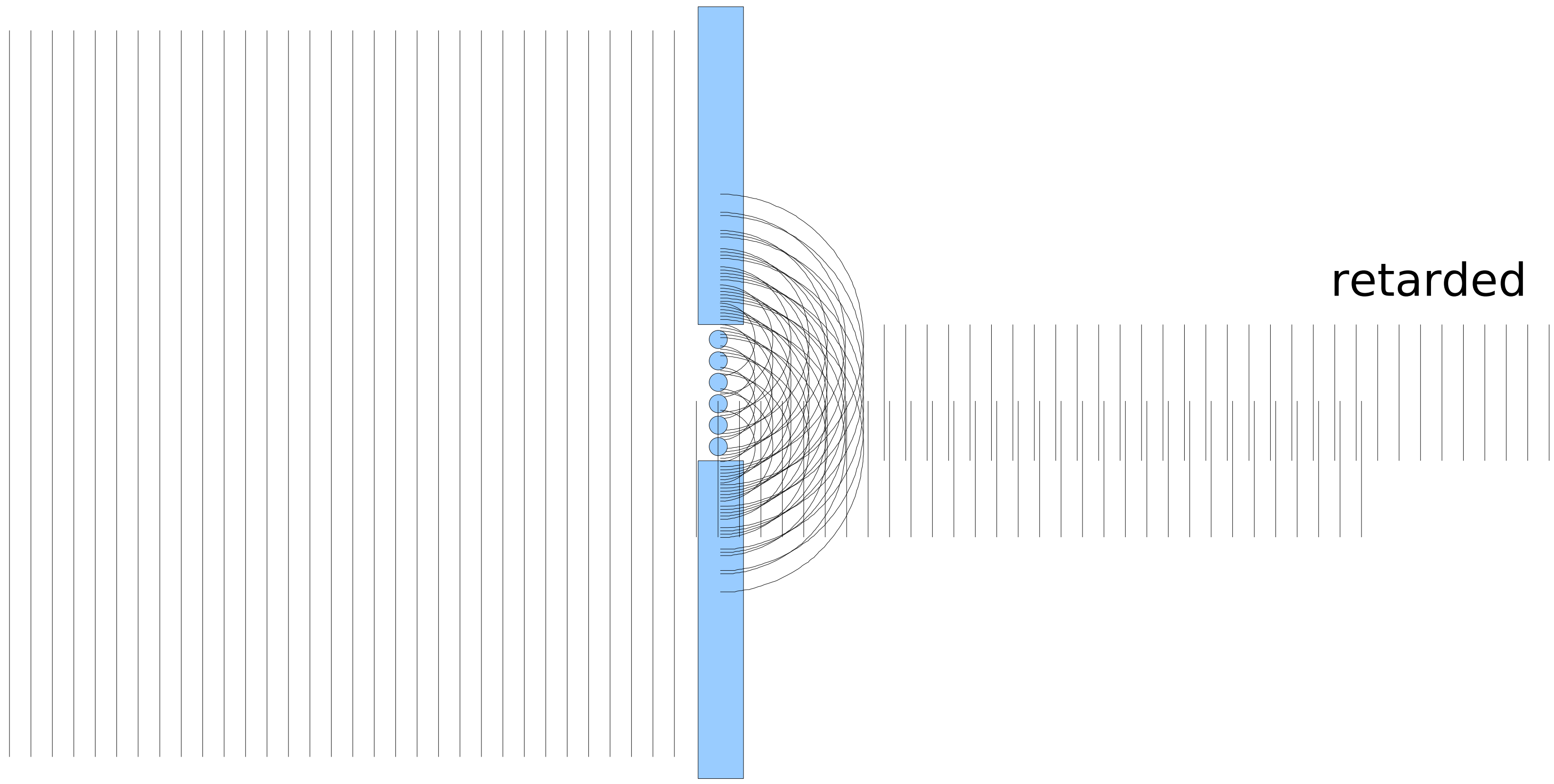

Single Slit Experiment

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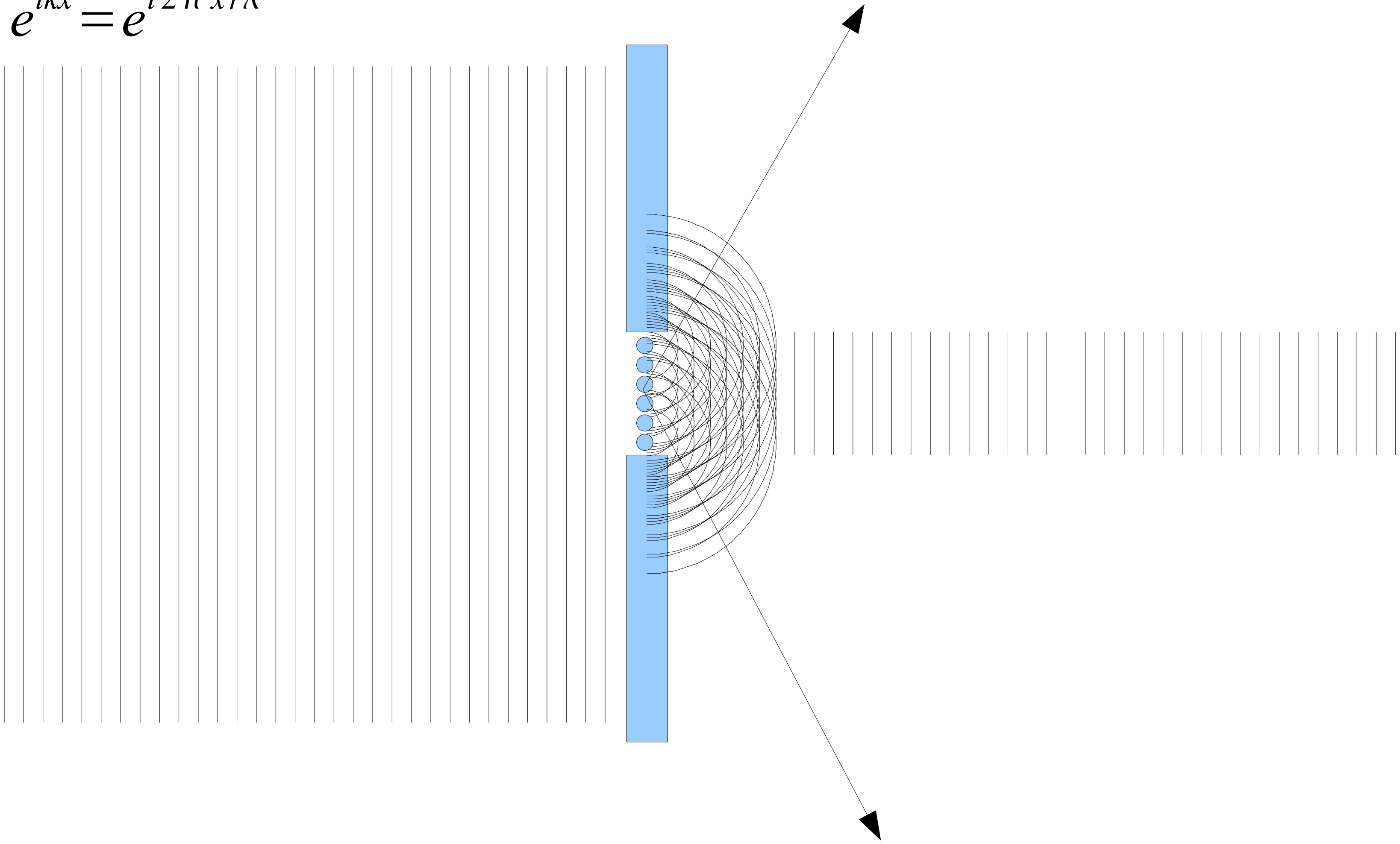
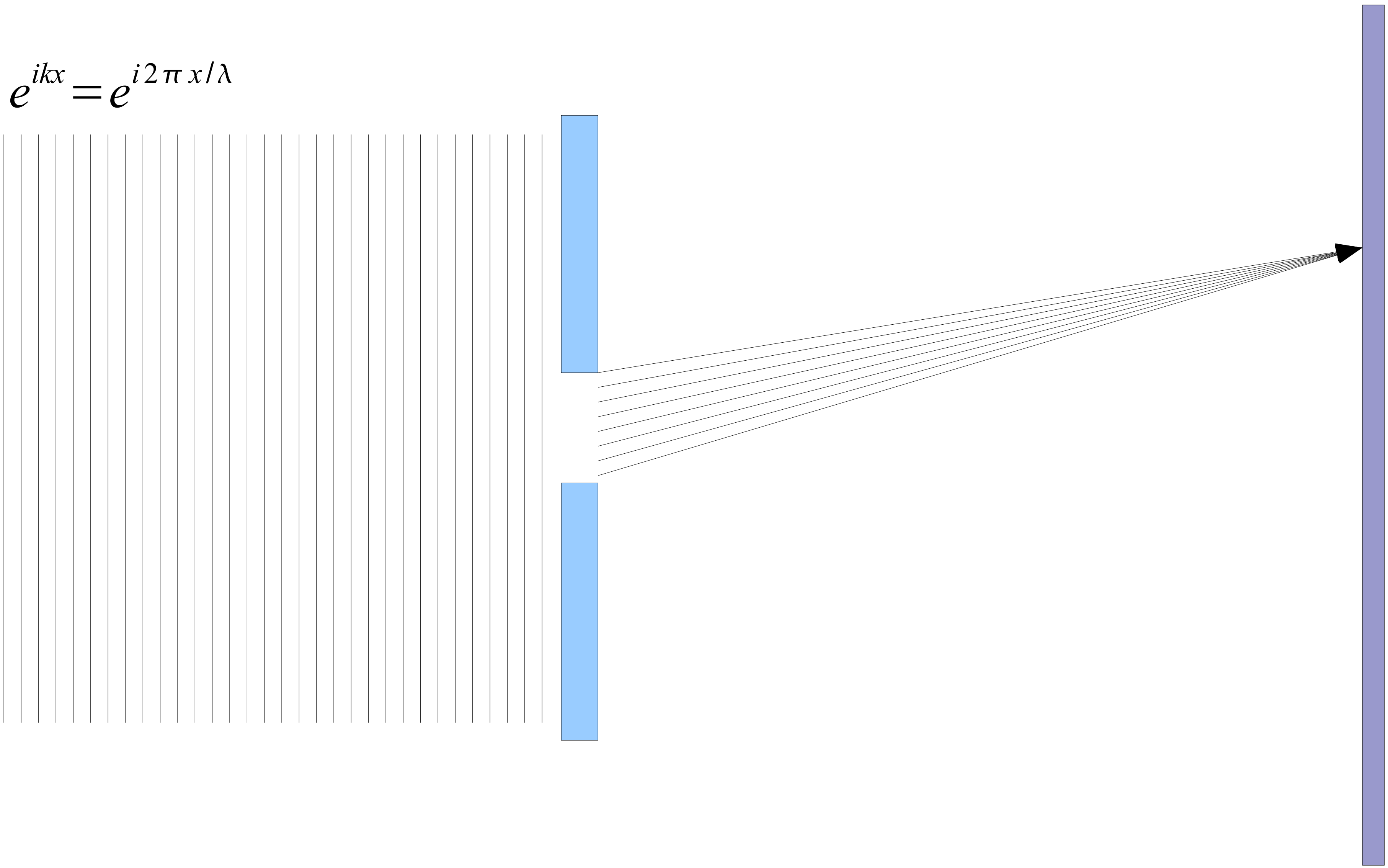
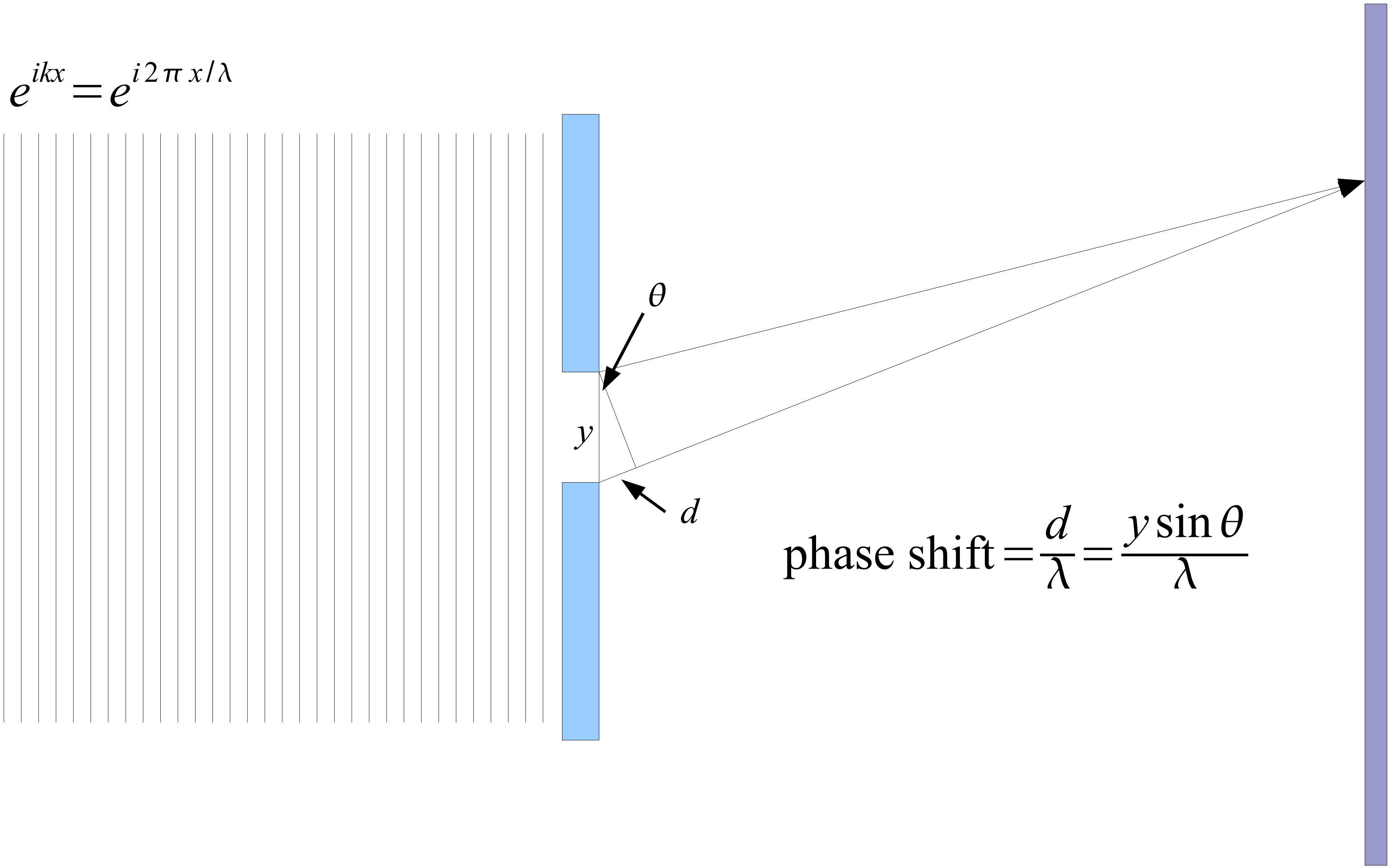


Single Slit Experiment

$$e^{ikx} = e^{i2\pi x/\lambda}$$



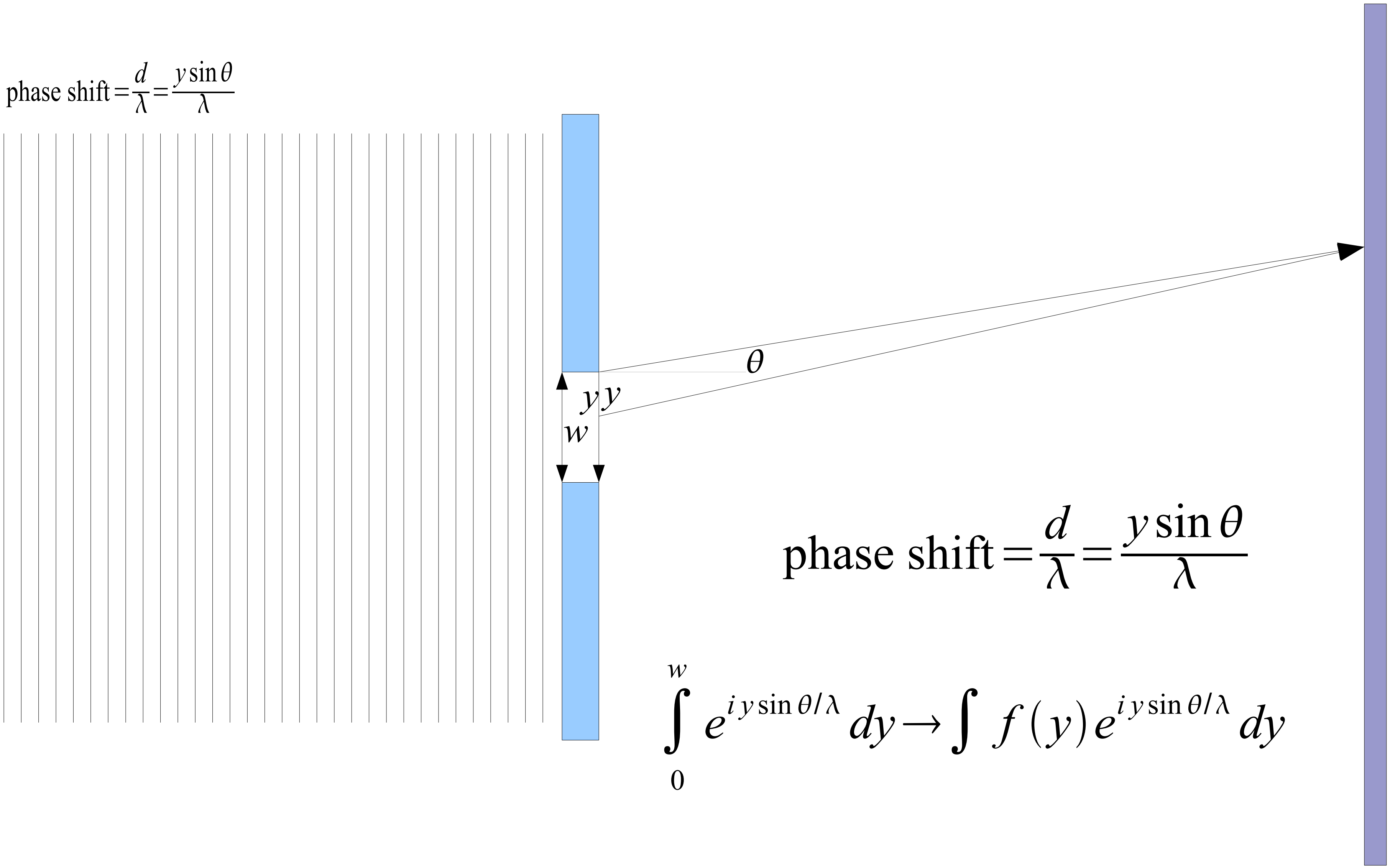
Single Slit Experiment



$$\text{phase shift} = \frac{d}{\lambda} = \frac{y \sin\theta}{\lambda}$$

Single Slit Experiment

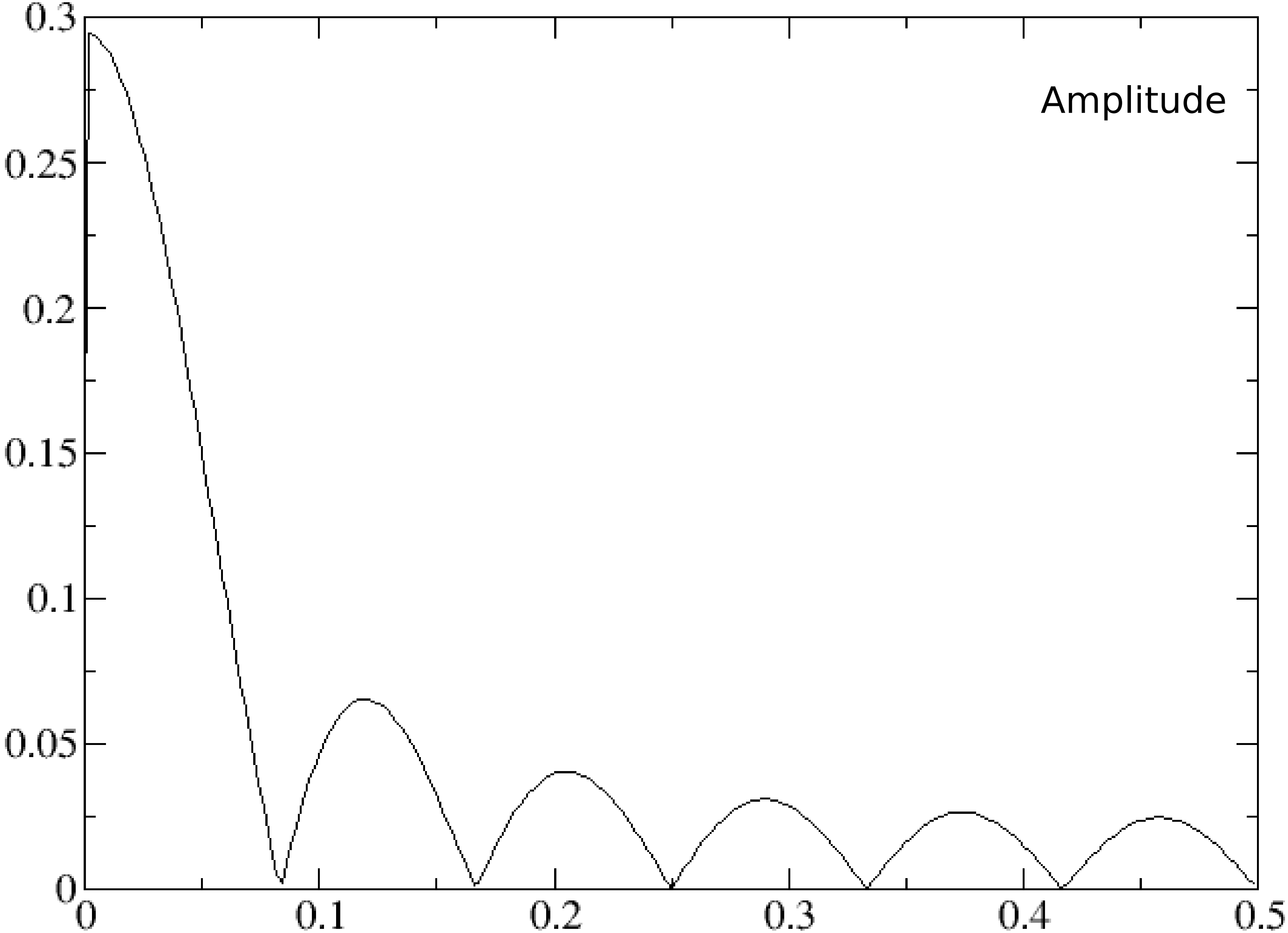
$$\text{phase shift} = \frac{d}{\lambda} = \frac{y \sin\theta}{\lambda}$$



$$\text{phase shift} = \frac{d}{\lambda} = \frac{y \sin\theta}{\lambda}$$

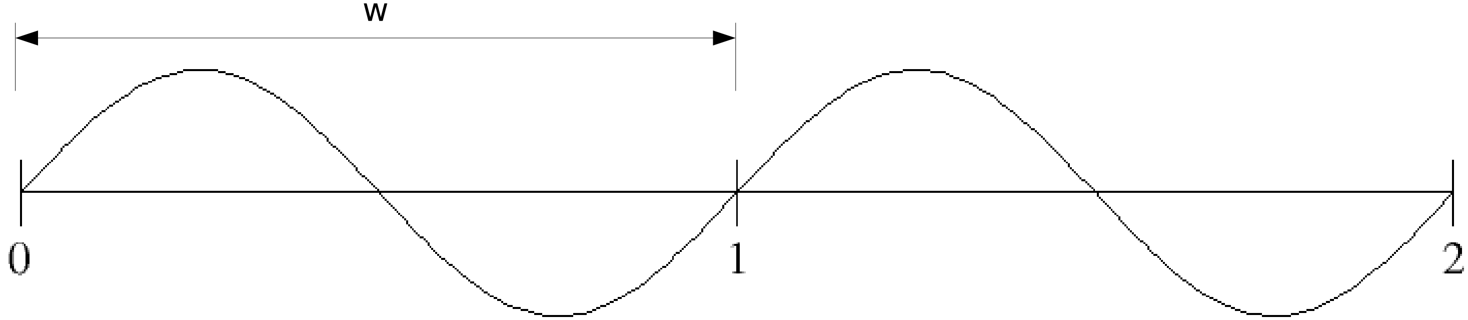
$$\int_0^w e^{iy\sin\theta/\lambda}\,dy \rightarrow \int f(y)\,e^{iy\sin\theta/\lambda}\,dy$$

FFT of a Square Pulse



Finite Fourier Transforms

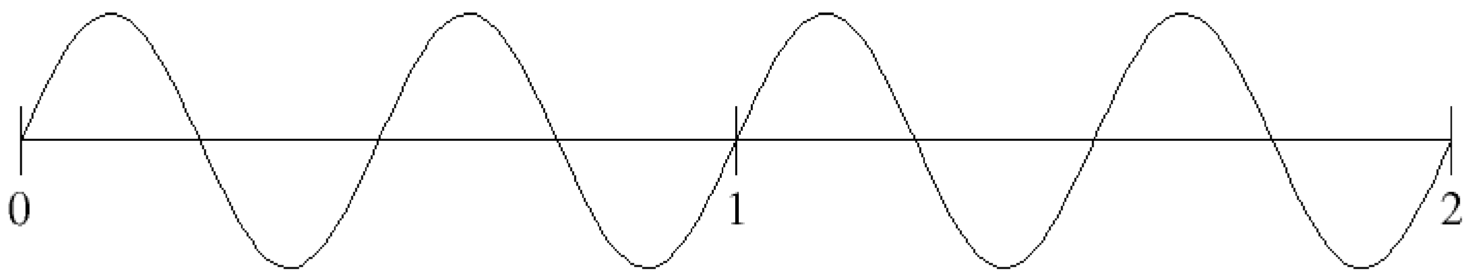
$$\bar{F}_k = \int_0^w f(x)\, e^{-i(2\pi k/w)x}\, dx$$



k=1

Finite Fourier Transforms

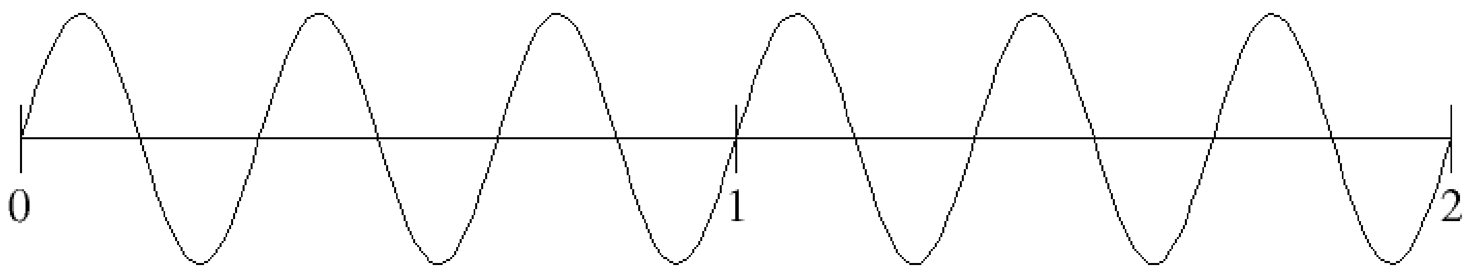
$$\bar{F}_k = \int_0^w f(x) e^{-i(2\pi k/w)x} dx$$



k=2

Finite Fourier Transforms

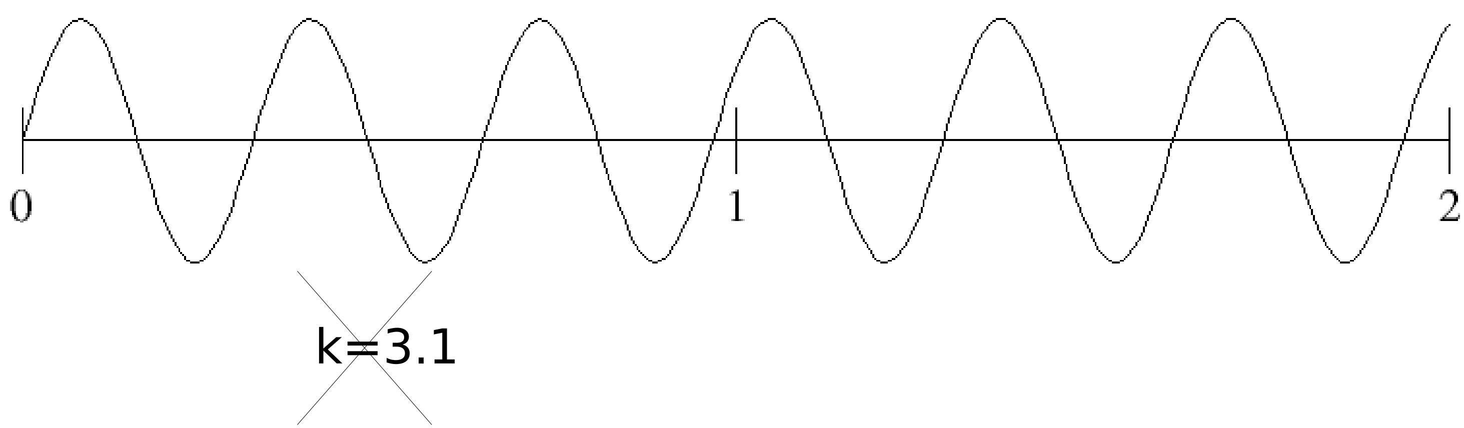
$$\bar{F}_k = \int_0^w f(x) e^{-i(2\pi k/w)x} dx$$



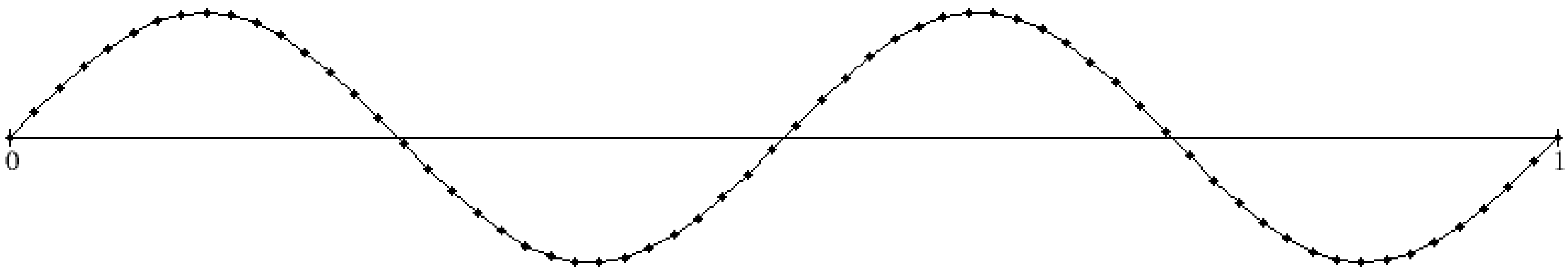
k=3

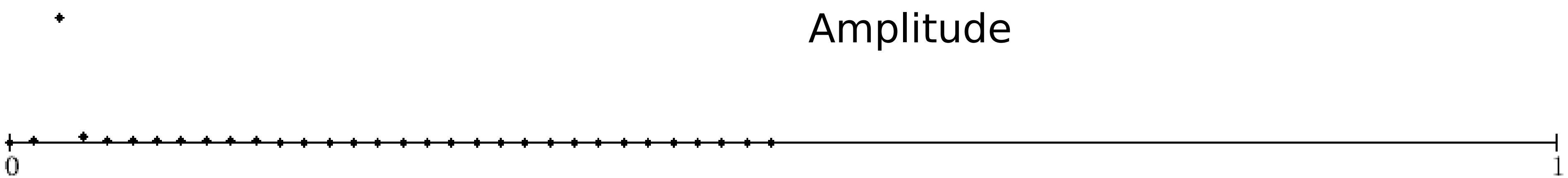
Finite Fourier Transforms

$$\bar{F}_k = \int_0^w f(x) e^{-i(2\pi k/w)x} dx$$



Finite Discrete Fourier Transforms



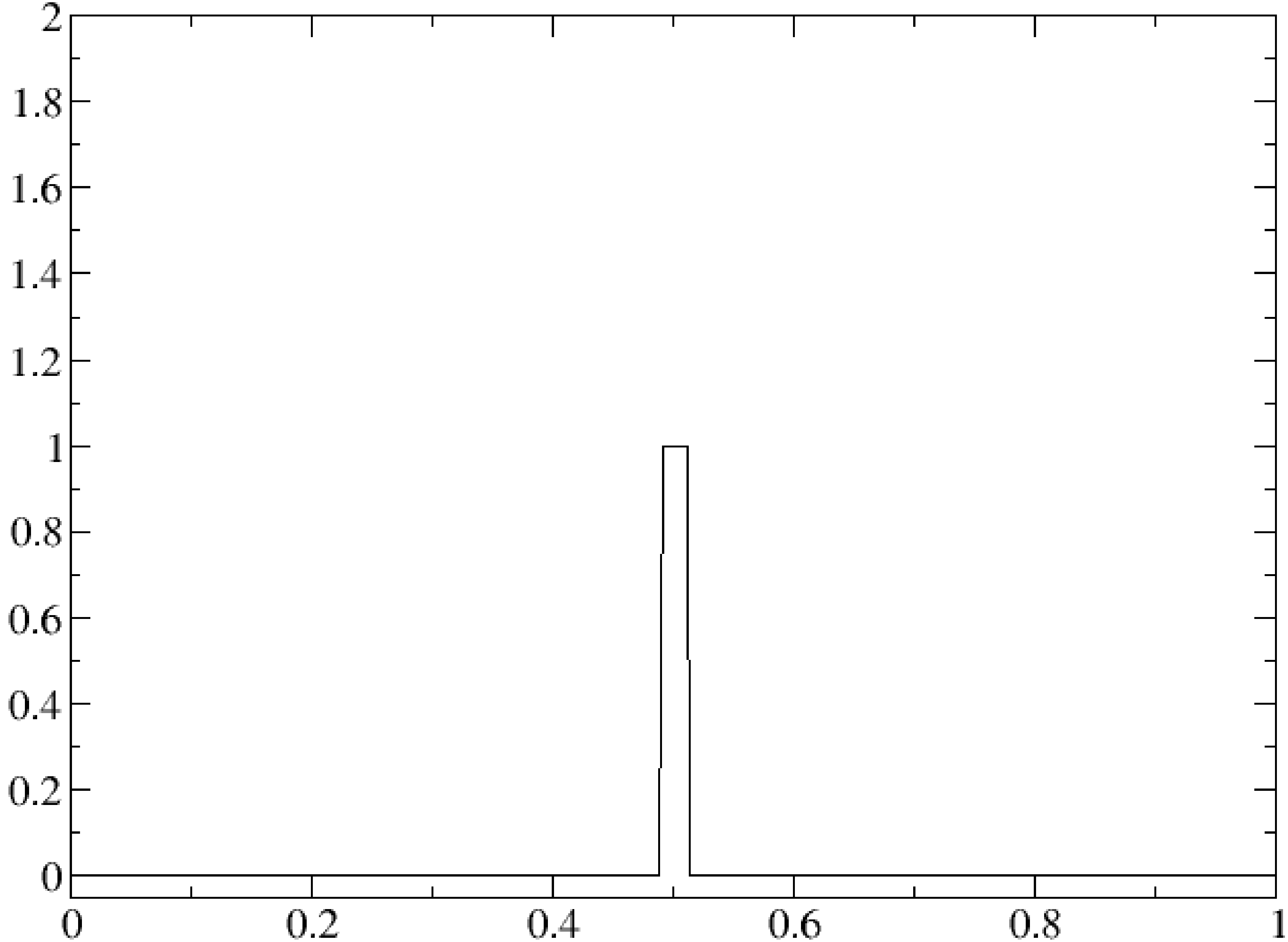


+phase

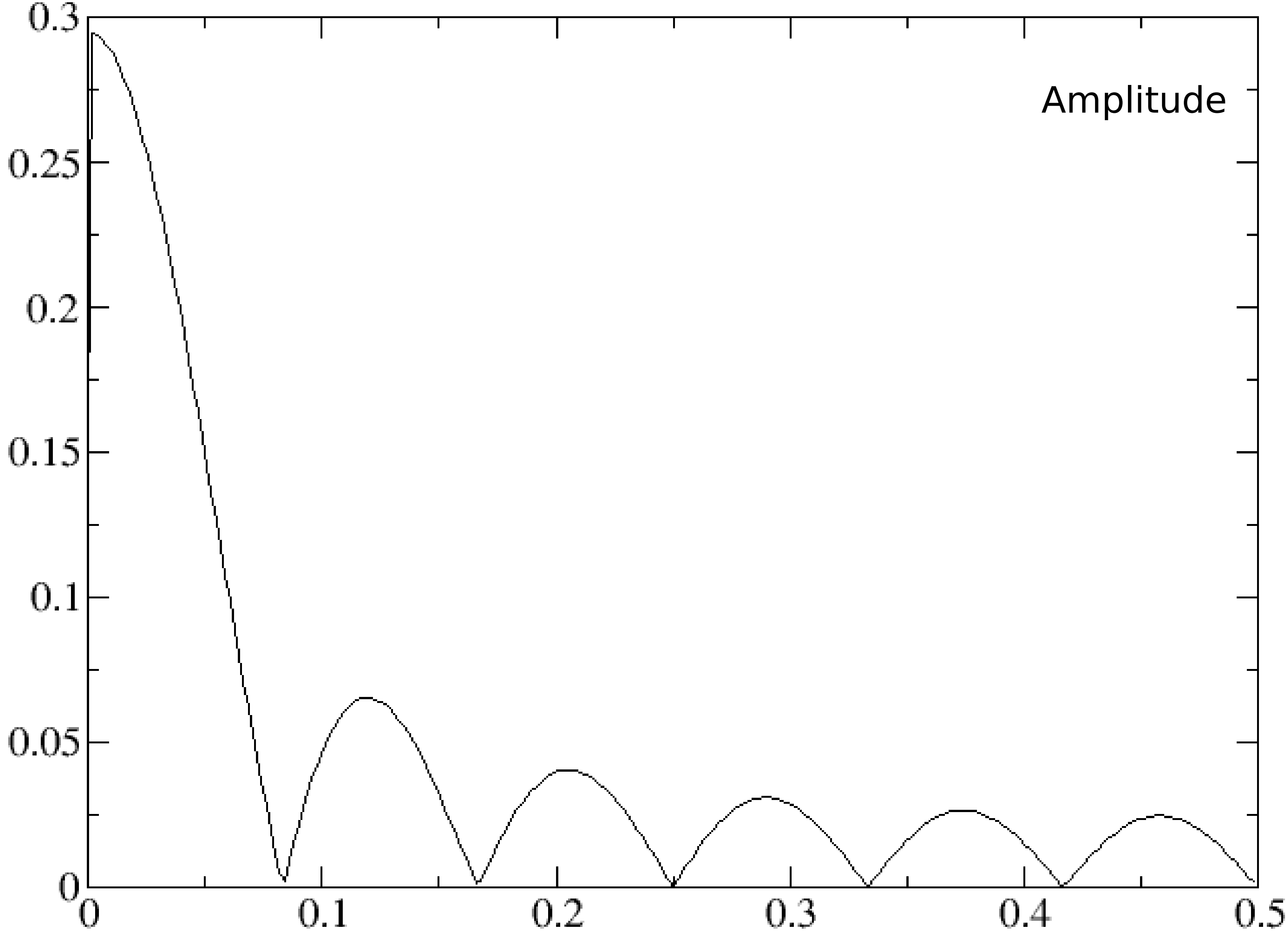
$$\bar{F}_k = \sum_{x=0}^{w} f(x) e^{-i(2\pi k/w)x}$$

Data stored as:
RIRIRIRIRIRIRI... -or-
APAPAPAPAPAPAP...

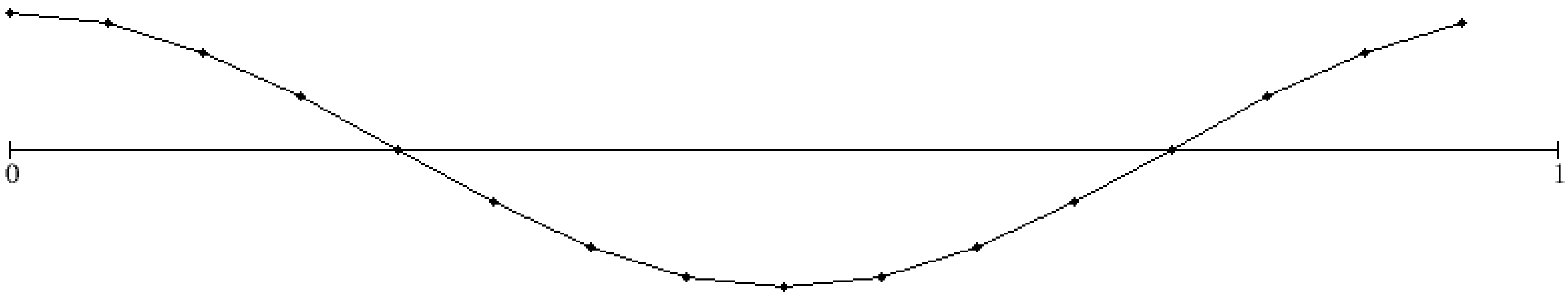
FFT of a Square Pulse



FFT of a Square Pulse



FFT Basis

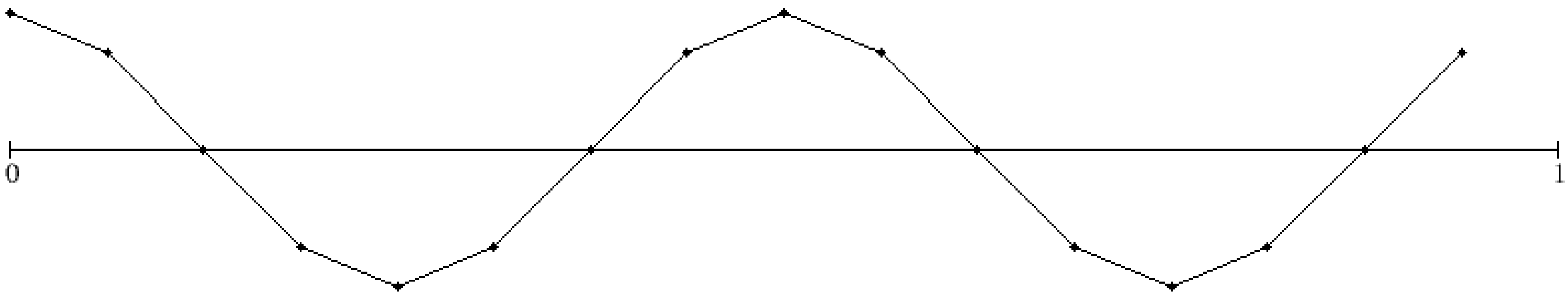


$\bar{F}(k) \rightarrow$ 0 , 0 , 1 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 FFT of curve above

	R	I	R	I	R	I	R	I	R	I	R	I	R	I	R	I	R	I
$k \rightarrow$	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8

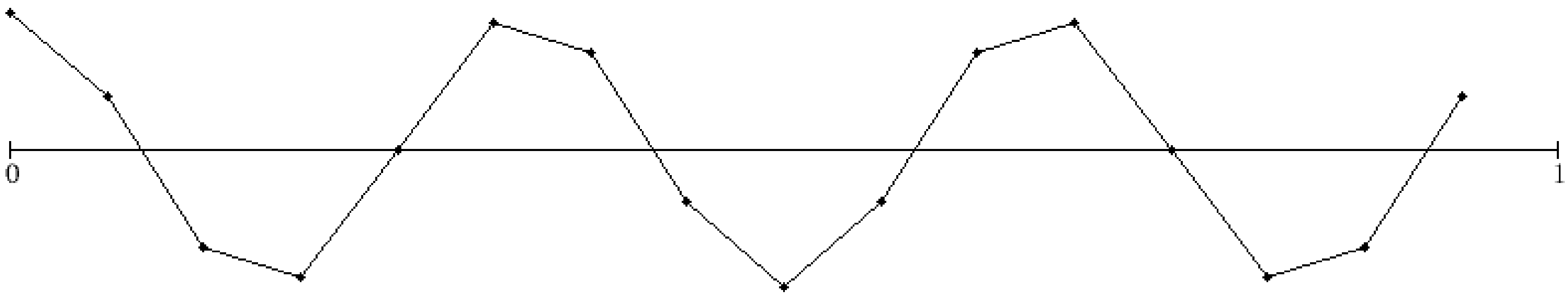
$$f_x = \sum_{k=0}^{n_k} \bar{F}(k) e^{i2\pi kx} = \cos 2\pi x$$

FFT Basis



0 ,	0 ,	0 ,	0 ,	1 ,	0 ,	0 ,	0 ,	0 ,	0 ,	0 ,	0 ,	0 ,	0 ,	0 ,	0 ,	0 ,	0	FFT of curve above
R	I	R	I	R	I	R	I	R	I	R	I	R	I	R	I	R	I	
0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	

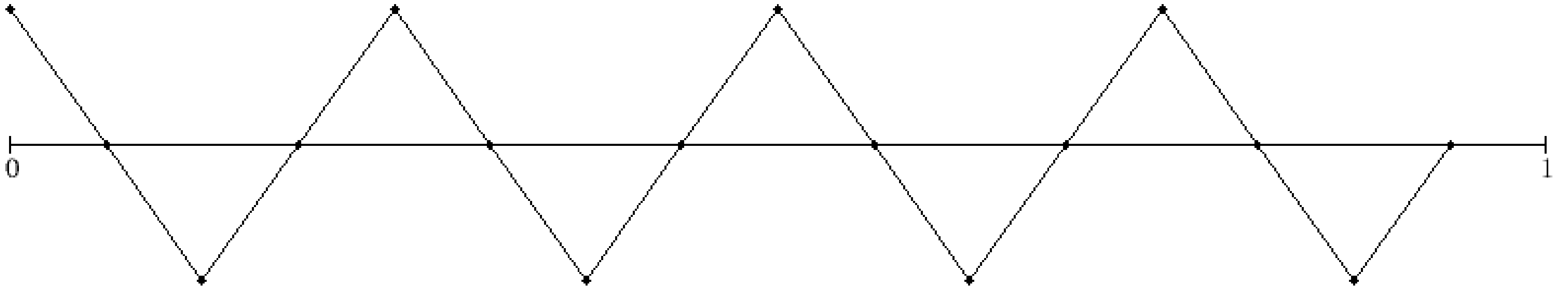
FFT Basis



0 , 0 , 0 , 0 , 0 , 0 , 1 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0   FFT of curve above

R	I	R	I	R	I	R	I	R	I	R	I	R	I	R	I	R	I
0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8

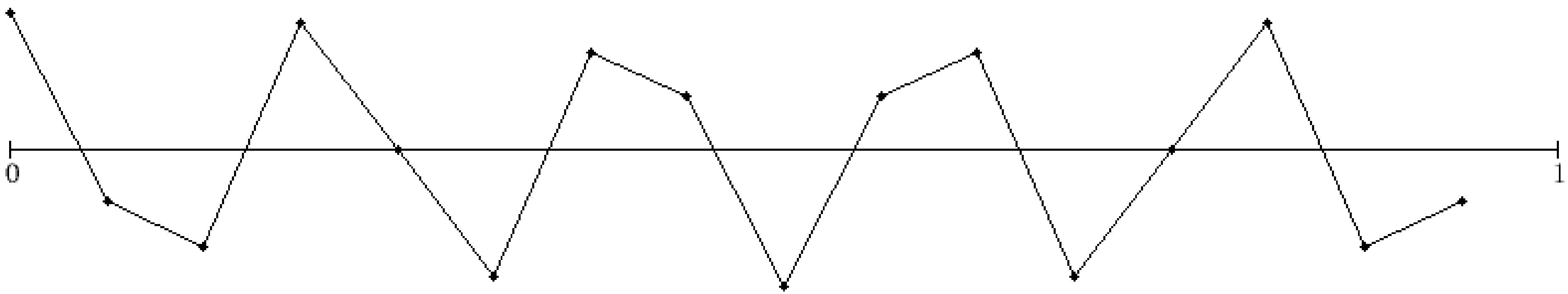
FFT Basis



0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 1 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0   FFT of curve above

R I R I R I R I R I R I R I R I R I
0 0 1 1 2 2 3 3 4 4 5 5 6 6 7 7 8 8

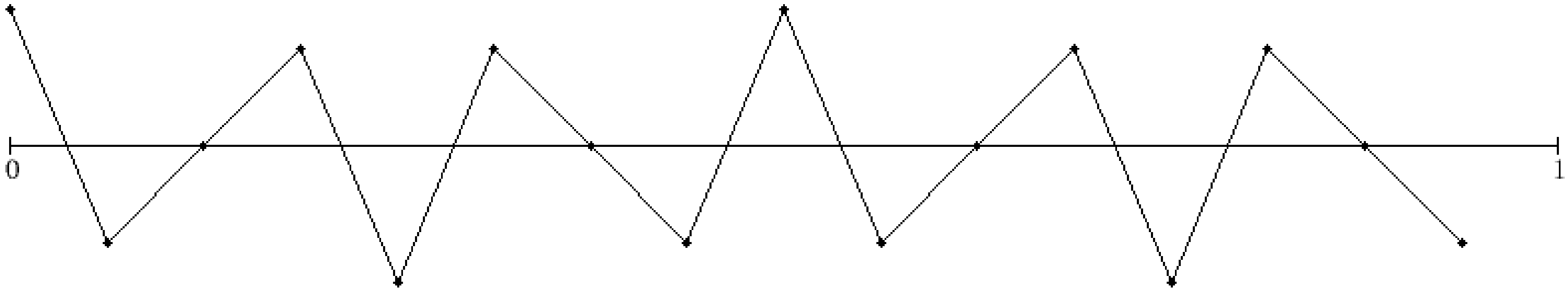
FFT Basis



0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 1 , 0 , 0 , 0 , 0 , 0 , 0 , 0   FFT of curve above

R I R I R I R I R I R I R I R I R I
0 0 1 1 2 2 3 3 4 4 5 5 6 6 7 7 8 8

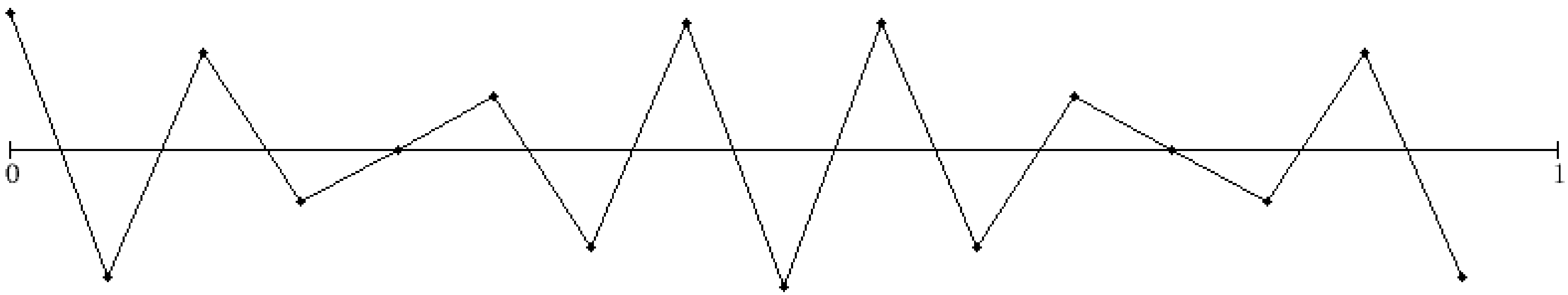
FFT Basis



0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 1 , 0 , 0 , 0 , 0 , 0   FFT of curve above

R	I	R	I	R	I	R	I	R	I	R	I	R	I	R	I	R	I
0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8

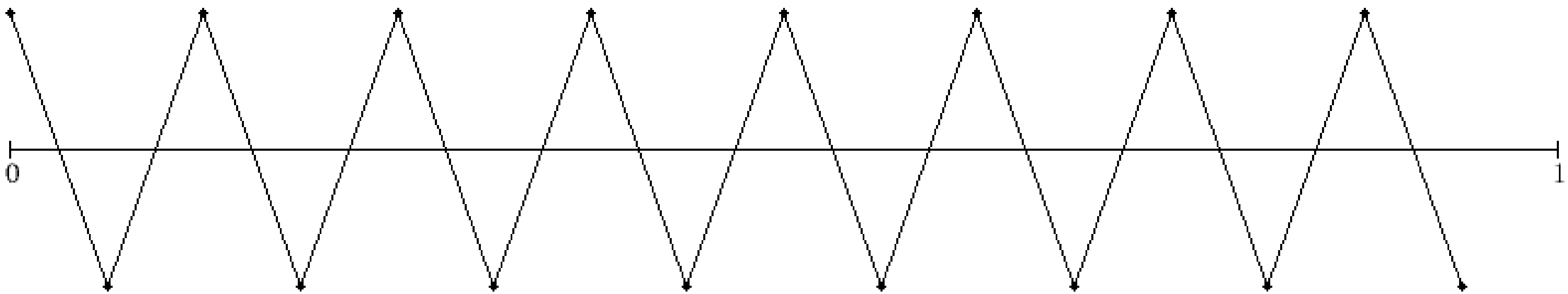
FFT Basis



0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 1 , 0 , 0 , 0   FFT of curve above

R	I	R	I	R	I	R	I	R	I	R	I	R	I	R	I	R	I
0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8

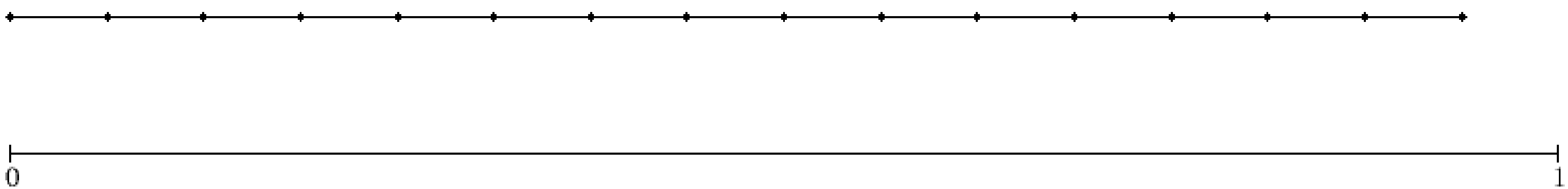
FFT Basis



0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 1 , 0 FFT of curve above

R	I	R	I	R	I	R	I	R	I	R	I	R	I	R	I	R	I
0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8

FFT Basis

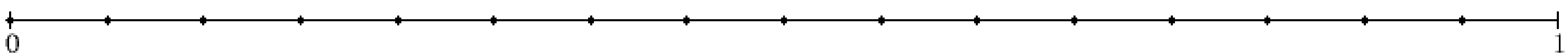


1 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0 , 0   FFT of curve above

R	I	R	I	R	I	R	I	R	I	R	I	R	I	R	I	R	I
0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8

$$\cos(kx), k = 0 \rightarrow 1$$

FFT Basis



0,	1,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	1	FFT of curve above
R	I	R	I	R	I	R	I	R	I	R	I	R	I	R	I	R	I	
0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	

$$\sin(2\pi k x), k=0 \rightarrow 0$$

$$\sin(2\pi k x), k=8, x=\frac{1}{16,}\frac{2}{16,}\ldots\frac{15}{16} \rightarrow 0$$

$$\bar{F}_k = \sum_{x=0}^{w} f(x)\, e^{-ikx} \rightarrow$$

$$\begin{bmatrix} \bar{F}_0 & \bar{F}_1 & \bar{F}_2 & \bar{F}_3 \end{bmatrix} = \begin{bmatrix} e^{-ik_0x_0} & e^{-ik_0x_1} & e^{-ik_0x_2} & e^{-ik_0x_3} \\ e^{-ik_1x_0} & e^{-ik_1x_1} & e^{-ik_1x_2} & e^{-ik_1x_3} \\ e^{-ik_2x_0} & e^{-ik_2x_1} & e^{-ik_2x_2} & e^{-ik_2x_3} \\ e^{-ik_3x_0} & e^{-ik_3x_1} & e^{-ik_3x_2} & e^{-ik_3x_3} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

note: conceptually interesting, but very inefficient. This is $O(n^2)$ whereas the FFT is O(n *log* n)

Fourier Transform Theorems

$$\text{if } f(x) \text{ real} \rightarrow \bar{F}(k) = \bar{F}^{*}(-k)$$

$$Convolution: f * g \rightarrow \bar{F}(k)\,\bar{G}(k)$$

$$Correlation: \int_{-\infty}^{\infty} g(x+a)\,h(a)\,da \rightarrow \bar{G}(k)\,\bar{H}^{*}(k)$$

$$Translation: f(x+x_0) \rightarrow \bar{F}(k)\,e^{ikx_0}$$

Homework

- Compute the continuous, infinite Fourier Transform of:

$$f(x) = \begin{matrix} 0 < x < 1 : f(x) = 1.0 \\ otherwise : f(x) = 0 \end{matrix}$$

- Now say the function is finite, defined on the range $0<x<2$. The transform is the same, of course, but the result is only defined for specific k values (see the lecture notes). Plot the sum of the first 10 elements of this Fourier series on the range $0<x<6$. (turn in the plot) what do you observe ? What would be different if you used 100 elements ? What would be the same ?