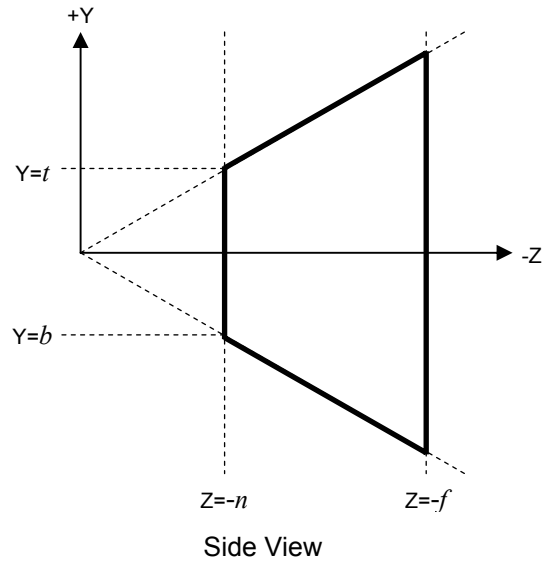
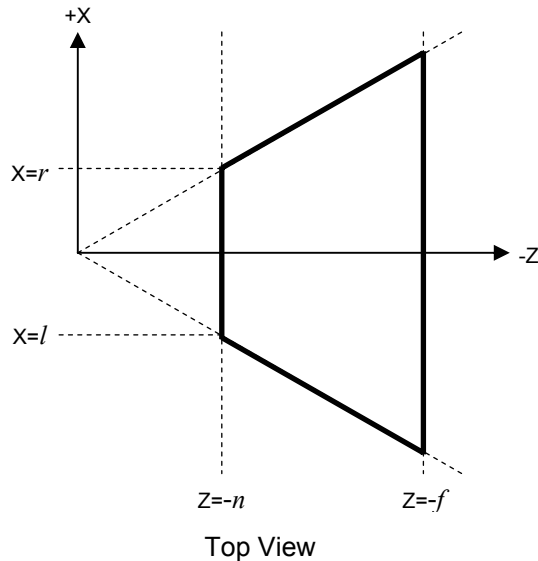


# Deriving the OpenGL Perspective Depth Transformation

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We define a viewing frustum in terms of the six planar boundaries defined by  $l$ ,  $r$ ,  $b$ ,  $t$ ,  $n$ , and  $f$  (represents left, right, bottom, top, near, and far planes respectively). The eye (or center of projection) is at the origin, the viewing direction is down the  $-Z$  axis, the projection or viewing plane is at  $z=-n$ , the near and far planes are defined as  $z=-n$  and  $z=-f$  respectively, the viewplane window is defined by rectangle with  $x$  extents  $[l,r]$  and  $y$  extents  $[b,t]$  on the  $z=-n$  plane, and the sides of the frustum are defined by the planes formed from the eye through the edges of the viewplane window.



The perspective depth transformation matrix is defined as from the geometry as follows (from the `glFrustum` call in OpenGL):

$$glFrustum(l, r, b, t, n, f) = M = \begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Our goal is to derive this matrix from the frustum geometry and the mapping between the eye-space points and normalized device coordinates (NDC or screen space after the perspective divide). The resulting points in NDC space are warped so that an orthographic projection will give the proper perspective viewing, lines will be preserved, and the resulting z values will give the proper visibility relationships. We can transform a 3d point in eye space by M to obtain a point in 4D homogeneous (or clipping) space, and then perform a normalization by dividing all components by w to obtain a point in NDC coordinates:

$$M \cdot \begin{bmatrix} eye_x \\ eye_y \\ eye_z \\ 1 \end{bmatrix} = \begin{bmatrix} clip_x \\ clip_y \\ clip_z \\ clip_w \end{bmatrix}$$

For eye points inside of the frustum:

$$clip_x, clip_y, clip_z \in [-clip_w, clip_w]$$

If we perform the perspective divide (homogeneous normalization), we obtain NDC coordinates:

$$\begin{bmatrix} clip_x / clip_w \\ clip_y / clip_w \\ clip_z / clip_w \\ clip_w / clip_w \end{bmatrix} = \begin{bmatrix} NDC_x \\ NDC_y \\ NDC_z \\ 1 \end{bmatrix}$$

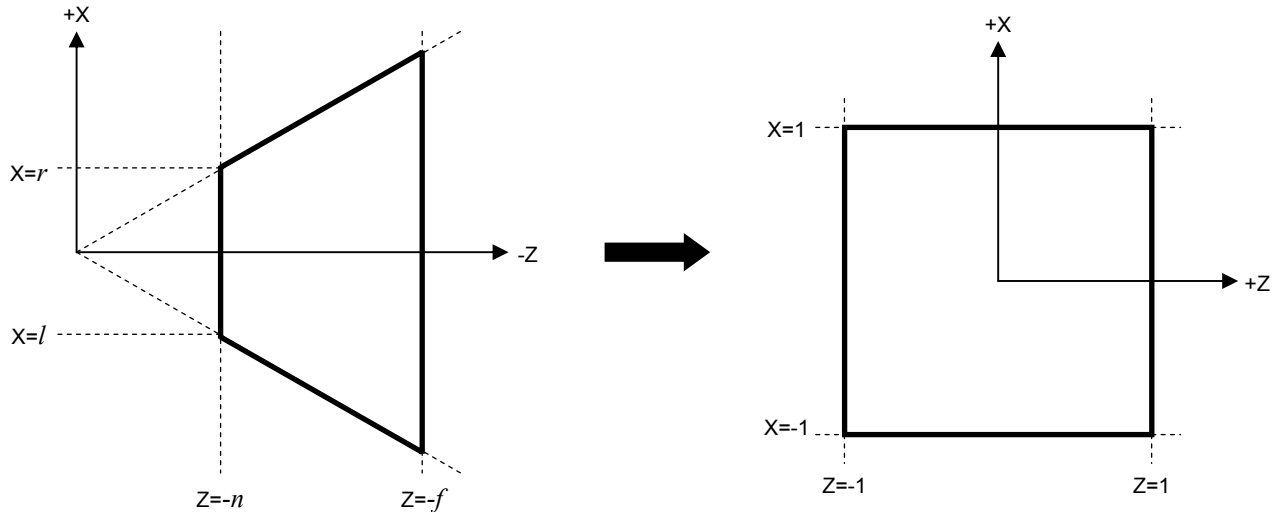
For points inside the frustum:

$$NDC_x, NDC_y, NDC_z \in [-1, 1]$$

This means that all points in the frustum in eye space map to the unit cube in NDC space defined between [-1, 1] for x, y, and z components after the perspective depth transformation (perspective transformation and perspective divide). More specifically, we have the following mapping:

Points in eye space	Points in NDC space
Left and right planes	x=-1 and x=1 planes respectively
Bottom and top planes	y=-1 and y=1 planes respectively
Near and far planes	z=-1 and z=1 planes respectively

Note that in eye space, we are looking down the  $-Z$  axis, but in NDC space we are looking down the  $+Z$  axis. The  $Z$  coordinates become negated in the perspective depth transformation:



Transformation from eye space to NDC space warps points in the perspective frustum to a rectangular, normalized viewing frustum where lines and depth values are preserved, and a simple orthographic projection will give the proper view.

We will first combine the matrix multiplication and perspective division into one equation for each NDC coordinate as a function of the input eye coordinates. We will then show how to derive each one of these equations from the frustum geometry, the desired mapping between the frustum points and the NDC unit cube, and the following properties of the perspective depth transformation:

- The  $x$  and  $y$  NDC coordinates will be the normalized coordinates of the eye space point projected onto the viewplane (near plane).
- The  $z$  coordinate in NDC space will be transformed such that lines in eye space are transformed into lines in NDC space. This is needed in order to allow for linear interpolation in screen space of depth values during polygon rasterization that will preserve the proper visibility ordering.

Let's first write the equations for each component directly from the perspective depth transformation and the perspective divide:

$$\begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} eye_x \\ eye_y \\ eye_z \\ 1 \end{bmatrix} = \begin{bmatrix} clip_x \\ clip_y \\ clip_z \\ clip_w \end{bmatrix}$$

We can rewrite this as four separate equations, one for each homogeneous (clip) component:

$$\begin{aligned}
 clip_x &= \frac{2n}{r-l} \cdot eye_x + \frac{r+l}{r-l} \cdot eye_z \\
 clip_y &= \frac{2n}{t-b} \cdot eye_y + \frac{t+b}{t-b} \cdot eye_z \\
 clip_z &= \frac{-(f+n)}{f-n} \cdot eye_z + \frac{-2fn}{f-n} \\
 clip_w &= -eye_z
 \end{aligned}$$

We can perform the perspective divide to obtain NDC coordinates as a function of eye coordinates:

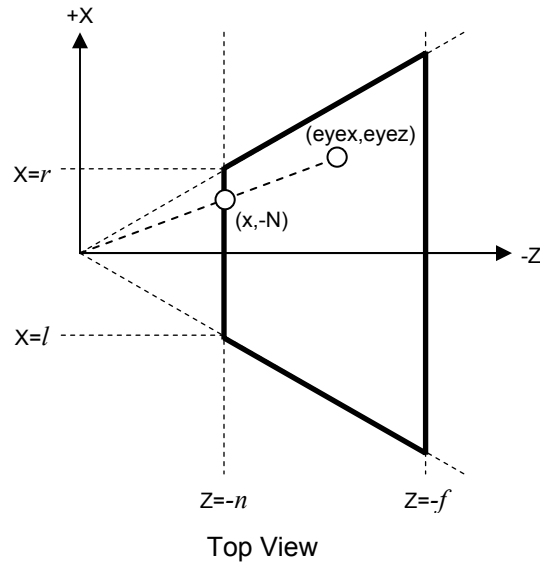
$$\begin{bmatrix} NDC_x \\ NDC_y \\ NDC_z \\ 1 \end{bmatrix} = \begin{bmatrix} clip_x / clip_w \\ clip_y / clip_w \\ clip_z / clip_w \\ clip_w / clip_w \end{bmatrix} = \begin{bmatrix} \frac{2n}{r-l} \cdot eye_x + \frac{r+l}{r-l} \cdot eye_z & / & -eye_z \\ \frac{2n}{t-b} \cdot eye_y + \frac{t+b}{t-b} \cdot eye_z & / & -eye_z \\ \frac{-(f+n)}{f-n} \cdot eye_z + \frac{-2fn}{f-n} & / & -eye_z \\ 1 & / & -eye_z \end{bmatrix}$$

Now we need only focus on each individual equation for the NDC components. If we can obtain these equations from the properties we mentioned above, then we have effectively derived the perspective depth transformation matrix.

For the NDC x coordinate, we need to derive the following equation:

$$NDC_x = \frac{\frac{2n}{r-l} \cdot eye_x + \frac{r+l}{r-l} \cdot eye_z}{-eye_z}$$

We can derive this from our frustum geometry by projecting the eye space point onto the viewplane:



We can find the projected x value using similar triangles, and then find the NDC x coordinate by normalizing between  $x=l$  and  $x=r$ :

$$\frac{x}{n} = \frac{eyex}{-eyez} \text{ which we can solve for } x \text{ to give } x = \frac{n \cdot eyex}{-eyez}$$

The resulting x value is in  $[l, r]$ . We must scale and translate to map  $[l, r]$  to  $[-1, 1]$ :

$$\begin{aligned} x &\in [l, r] \\ \frac{x-l}{r-l} &\in [0, 1] \\ NDC_x &= \frac{x-l}{r-l} \cdot 2 - 1 \in [-1, 1] \end{aligned}$$

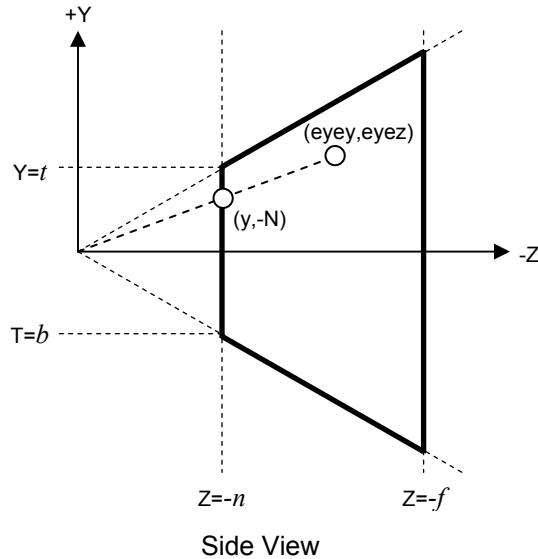
Now we can substitute our x value in terms of the original eye x coords to obtain the final form of the equation:

$$\begin{aligned} NDC_x &= \frac{\frac{n \cdot eyex}{-eyez} - l}{r-l} \cdot 2 - 1 = \frac{\frac{2n \cdot eyex}{-eyez} - 2l}{r-l} - 1 = \frac{2n \cdot eyex}{(r-l) \cdot -eyez} - \frac{2l}{r-l} - \frac{r-l}{r-l} \\ &= \frac{2n \cdot eyex}{(r-l) \cdot -eyez} - \frac{r+l}{r-l} = \frac{\frac{2n}{r-l} \cdot eyex + \frac{r+l}{r-l} \cdot eyez}{-eyez} \end{aligned}$$

This gives us what we need:

$$NDC_x = \frac{\frac{2n}{r-l} \cdot eyex + \frac{r+l}{r-l} \cdot eyez}{-eyez}$$

We can obtain the NDC y coordinate using the same strategy. The only difference will be the replacement of  $l$  and  $r$  with  $b$  and  $t$  respectively, and the use of eye's y coordinate in place of the x coordinate.



This will give us the second equation:

$$NDC_y = \frac{\frac{2n}{t-b} \cdot eye_y + \frac{t+b}{t-b} \cdot eye_z}{-eye_z}$$

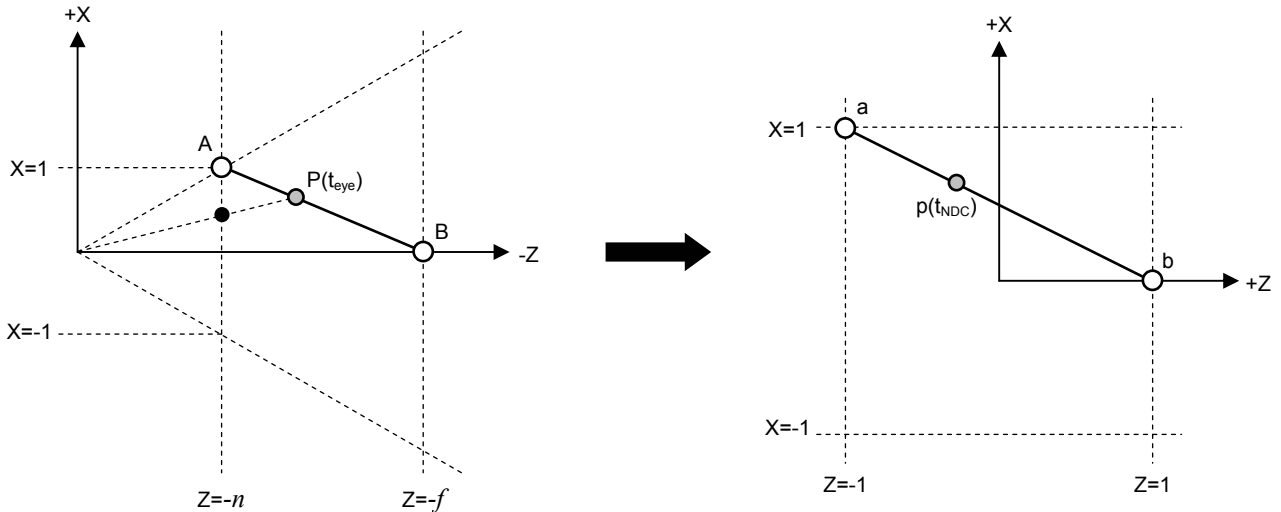
The most interesting and complex NDC coordinate equation is for the z component. We will use our knowledge of the projection of the points and the preservation of lines in NDC space to derive the following equation:

$$NDC_z = \frac{\frac{-(f+n)}{f-n} \cdot eye_z + \frac{-2fn}{f-n}}{-eye_z}$$

Here is the strategy we will take:

- Choose a line segment in eye space that is easy to characterize and write a parametric equation for the line between its endpoints as a function of  $t_{eye}$
- Parameterize the same line in NDC space as a function of  $t_{NDC}$
- Project a general point on the eye space line to the viewplane and use this equation to relate  $t_{eye}$  to  $t_{NDC}$
- Solve for  $t_{eye}$  as a function of  $t_{NDC}$
- Use this function to write an equation for a point along the line in NDC space as a function of  $t_{eye}$  instead of  $t_{NDC}$
- Solve for the NDC point's z value in terms of  $t_{NDC}$
- Write  $t_{NDC}$  as a function of the eye space point's z value
- Use previous two equations to solve for the NDC point's z value in terms of eye space point's z value. This is what we want!

First we will show the frustum geometry in eye space and in NDC space, the chosen line endpoints and their mapping between eye and NDC space (AB in eye space, ab in NDC space), and an arbitrary linearly interpolated point along each line (P in eye space, p in NDC space):



The choice of line segment endpoints is somewhat arbitrary. We chose this configuration to simplify the derivation of the z coordinate equation. In addition, we simply assume a symmetric frustum with viewplane window boundaries  $(l,r,b,t)=(-1,1,-1,1)$ . These simplifications do not change the generality of the final z equation since the final result does not depend on these values. From the figure above we have the following equations:

Eye Space		NDC (Screen) Space	
$A_x=1$	$B_x=0$	$a_x=1$	$b_x=0$
$A_z=-n$	$B_z=-f$	$a_z=-1$	$b_z=1$
$P(t_{eye})=A + (B-A) \cdot t_{eye}$		$p(t_{NDC})=a + (b-a) \cdot t_{NDC}$	

The chosen endpoints in eye space have an obvious mapping to NDC space. We need only find the z equation that preserves the lines between the endpoints in both spaces. We first need to relate the two parametric values t. We can obtain an equation containing both by simply applying the projection to the point  $P(t_{eye})$  and finding its x value in NDC space. Since we have chosen a simple symmetric normalized viewing frustum the projection can be found using similar triangles (as shown before for the x coordinate equation derivation):

$$p_x(t_{NDC}) = \frac{n \cdot P_x(t_{eye})}{-P_z(t_{eye})}$$

We can then substitute the full equation for the interpolated points and solve for  $t_{NDC}$  in terms of  $t_{eye}$ :

$$a_x + (b_x - a_x) \cdot t_{NDC} = \frac{n \cdot (A_x + (B_x - A_x) \cdot t_{eye})}{-(A_z + (B_z - A_z) \cdot t_{eye})}$$

To simplify the problem, we can substitute all of our known values from the table:

$$1 + (0 - 1) \cdot t_{NDC} = \frac{n \cdot (1 + (0 - 1) \cdot t_{eye})}{-(-n + (-f + n) \cdot t_{eye})}$$

Now we can easily solve for  $t_{NDC}$ :

$$1 - t_{NDC} = \frac{n - n \cdot t_{eye}}{n - (n - f) \cdot t_{eye}}$$

$$t_{NDC} = \frac{f \cdot t_{eye}}{n - (n - f) \cdot t_{eye}}$$

We can now use this function to write an equation for an interpolated point's z value along the line in NDC space as a function of  $t_{eye}$  instead of  $t_{NDC}$ :

$$p_z(t_{NDC}) = a_z + (b_z - a_z) \cdot t_{NDC}$$

$$p_z\left(\frac{f \cdot t_{eye}}{n - (n - f) \cdot t_{eye}}\right) = a_z + (b_z - a_z) \cdot \frac{f \cdot t_{eye}}{n - (n - f) \cdot t_{eye}}$$

substitute known quantities for NDC space endpoint coordinates and simplify:

$$p_z(t_{eye}) = -1 + (1 + 1) \cdot \frac{f \cdot t_{eye}}{n - (n - f) \cdot t_{eye}}$$

$$p_z(t_{eye}) = \frac{-n + (n + f) \cdot t_{eye}}{n + (f - n) \cdot t_{eye}}$$

Our ultimate goal is to obtain the general equation for the NDC space z coordinate ( $p_z$ ) in terms of the eye space z coordinate ( $P_z$ ). We can use the interpolated point equation for an eye space point to solve for  $t_{eye}$  in terms of the eye space z coordinates:

$$P_z(t_{eye}) = A_z + (B_z - A_z) \cdot t_{eye}$$

$$t_{eye} = \frac{P_z - A_z}{B_z - A_z}$$

We can then use this equation to obtain NDC space z in terms of eye space z:

$$p_z\left(\frac{P_z - A_z}{B_z - A_z}\right) = \frac{-n + (n + f) \cdot \frac{P_z - A_z}{B_z - A_z}}{n + (f - n) \cdot \frac{P_z - A_z}{B_z - A_z}}$$



We can then substitute our known values for the eye space endpoints (A,B):

$$p_z(P_z) = \frac{-n + (n + f) \cdot \frac{P_z + n}{n - f}}{n + (f - n) \cdot \frac{P_z + n}{n - f}}$$

We can then simplify this equation to obtain the desired form for the z coordinate equation:

$$= \frac{\frac{-n \cdot (n - f)}{n - f} + \frac{f \cdot P_z + fn + n \cdot P_z + n^2}{n - f}}{n - (n - f) \cdot \frac{P_z + n}{n - f}}$$

$$= \frac{\frac{-n^2 + fn + f \cdot P_z + fn + n \cdot P_z + n^2}{n - f}}{-P_z}$$

$$= \frac{\frac{2fn + f \cdot P_z + n \cdot P_z}{n - f}}{-P_z}$$

$$= \frac{\frac{-2fn - f \cdot P_z - n \cdot P_z}{f - n}}{-P_z}$$

$$= \frac{\frac{-(f + n) \cdot P_z - 2fn}{f - n}}{-P_z}$$

Remember that  $p_z$  is the NDC space z ( $NDC_z$ ) and  $P_z$  is the eye space z ( $eye_z$ ), so we can then substitute to obtain the exact form from the perspective transformation matrix:

$$NDC_z = \frac{\frac{-(f + n)}{f - n} \cdot eye_z + \frac{-2fn}{f - n}}{-eye_z}$$

We have now completed the entire derivation of the perspective depth transformation matrix!

# OpenGL's Perspective Projection Matrix

(based on Kenny Hoff's <http://www.cs.unc.edu/~hoff/techrep/perspective.doc> )

OpenGL's perspective projection matrix  $M$  maps from 3d eyespace to 2d normalized device coordinates (NDC):

$$M \cdot \begin{bmatrix} eye_x \\ eye_y \\ eye_z \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} NDC_x \\ NDC_y \\ NDC_z \end{bmatrix}$$

The matrix is given by

$$\begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

The matrix has several properties, which we will utilize in its derivation.

1. The view frustum defined by the parameters (left, right, bottom, top, near, far) is mapped to the canonical view volume in (-1, 1, -1, 1, -1, 1).

Eyespace	NDC
left, right	-1, 1
bottom, top	-1, 1
near, far	-1, 1

2. The signs of the z-values are reversed. This is because in eyespace, the view is down the  $-z$ -axis, whereas in NDC, the view is down the  $+z$ -axis.

3. Lines and depth values are preserved for correct perspective viewing and the z-buffer algorithm.

4. Orthographic projection into device-dependent screen coordinates yields a perspective-correct view.

Before starting the derivation, write the linear equations that correspond to the matrix.

$$NDC_x = \frac{\frac{2n}{r-l} \cdot eye_x + \frac{r+l}{r-l} \cdot eye_z}{eye_z}$$

$$NDC_y = \frac{\frac{2n}{t-b} \cdot eye_y + \frac{t+b}{t-b} \cdot eye_z}{eye_z}$$

$$NDC_z = \frac{-\frac{f+n}{f-n} \cdot eye_z + \frac{-2fn}{f-n}}{f-n}$$

First, derive the equation for  $NDC_x$ .

**Step 1.** Project the  $eye_x$  onto the near plane  $z = -n$  using similar triangles.

$$x' = \frac{-eye_x}{eye_z} \cdot n$$

**Step 2.** Scale and translate the resulting coordinates to map them to the canonical view volume.

$$\frac{x'-l}{r-l} \in [0, 1] \text{ so } 2 \cdot \frac{x'-l}{r-l} - 1 \in [-1, 1]$$

Substituting the equation under Step 1. for  $x'$  in Step 2. above, we get

$$NDC_x = 2 \cdot \frac{x'-l}{r-l} - 1 = 2 \cdot \frac{\frac{-eye_x}{eye_z} \cdot n - l}{r-l} - 1 = \frac{\frac{2n}{r-l} \cdot eye_x + \frac{r+l}{r-l} \cdot eye_z}{-eye_z}$$

$$NDC_x = \frac{\frac{2n}{r-l} \cdot eye_x + \frac{r+l}{r-l} \cdot eye_z}{eye_z}$$

Note that the equation for  $NDC_y$  utilizes an identical derivation, so that  $eye_y$  replaces  $eye_x$ , and  $b$ ,  $t$  replace  $l$ ,  $r$ .

$$NDC_y = \frac{\frac{2n}{t-b} \cdot eye_y + \frac{t+b}{t-b} \cdot eye_z}{eye_z}$$

Now, derive the equation for  $NDC_z$ .

**Step 1.** Draw a line in eyespace and a corresponding line in NDC, such that their parametric equations are easy to determine. Set up a table of the lines' endpoint labels and values.

The parametric equations are easy to determine if (a) the endpoints of the lines have known values and (b) one of the endpoints is at  $x=0$ . So draw a line in eyespace (a) between the near plane and the far plane with (b) one of the endpoints at  $x=0$ .

Let  $A, B$  be the endpoints of the line in eyespace, and let  $a, b$  be the corresponding endpoints of the line in NDC. Let  $P$  be an arbitrary point on the line in eyespace, and let  $p$  be the corresponding point on the line in NDC. Let  $t_{eye}$  be the weight in the parametric equation for the line in eyespace, and let  $t_{NDC}$  be the weight of the parametric equation for the line in NDC.

Eyespace	NDC
$A_x=1$	$a_x=1$
$A_z=-n$	$a_z=1$
$B_x=0$	$b_x=0$
$B_z=-f$	$b_z=1$

Our goal is to find  $NDC_z(eye_z) = p_z(P_z)$ .

**Step 2.** Solve for  $NDC_z$  in terms of  $eye_z$  by substituting equations for  $eye_z$  and  $NDC_z$ , substituting values for the endpoints of the lines, and doing algebraic rearrangement.

The equations for the lines in eyespace and in NDC are

$$P(t_{eye}) = A + (B - A) \cdot t_{eye}$$

and

$$p(t_{NDC}) = a + (b - a) \cdot t_{NDC}$$

From the earlier derivation of  $NDC_x$  using similar triangles, we saw that, before mapping to the canonical view volume

$$p_x(t_{eye}) = \frac{n \cdot P_x(t_{eye})}{-P_z(t_{eye})}$$

Since  $p_x(t_{NDC}) = p_x(t_{eye})$ , then we can set the equations for these labels equal to each other to get

$$a_x + (b_x - a_x) \cdot t_{NDC} = n \cdot \frac{A_x + (B_x - A_x) \cdot t_{eye}}{-(A_z + (B_z - A_z) \cdot t_{eye})}$$

Substitute the values of the endpoints into their labels to get

$$0 + (1-0) \cdot t_{NDC} = n \cdot \frac{0 + (1-0) \cdot t_{eye}}{-(-n + (-f + n) \cdot t_{eye})}$$

Solve for  $t_{NDC}$  to get

$$t_{NDC} = \frac{f \cdot t_{eye}}{n - (n - f) \cdot t_{eye}}$$

Substitute the equation for  $t_{NDC}$  into the equation for  $p_z(t_{NDC})$  to get  $p_z(t_{eye})$

$$p_z(t_{eye}) = a_z + (b_z - a_z) \cdot \frac{f \cdot t_{eye}}{n - (n - f) \cdot t_{eye}}$$

Substitute the values for the endpoints into the above equation to get

$$p_z(t_{eye}) = -1 + (1+1) \cdot \frac{f \cdot t_{eye}}{n - (n - f) \cdot t_{eye}} = -1 + 2 \cdot \frac{f \cdot t_{eye}}{n - (n - f) \cdot t_{eye}} = \frac{-n + (n + f) \cdot t_{eye}}{n + (f - n) \cdot t_{eye}}$$

Remember that our goal is to find  $NDC_z(eye_z) = p_z(P_z)$ . Solve the equation for  $P_z(t_{eye})$  for  $t_{eye}$

$$t_{eye}(P_z) = \frac{P_z - A_z}{B_z - A_z}$$

...And substitute the equation for  $t_{eye}(P_z)$  into the equation for  $p_z(t_{eye})$

$$p_z(P_z) = \frac{-n + (n + f) \cdot \frac{P_z - A_z}{B_z - A_z}}{n + (f - n) \cdot \frac{P_z - A_z}{B_z - A_z}} = \frac{-(f + n) \cdot P_z - \frac{2fn}{f - n}}{P_z}$$

Since  $P_z = eye_z$  and  $p_z = NDC_z$ , then we have completed the derivation of the equation for  $NDC_z$ .

$$NDC_z = \frac{-\frac{f + n}{f - n} \cdot eye_z + \frac{-2fn}{f - n}}{f - n}$$